Global regularity for minimal sets near a union of two planes

Xiangyu Liang

Département de Mathématiques, Bât 425, Université Paris-Sud 11, 91405 Orsay Cedex
xiangyuliang@gmail.com

Abstract.

We discuss the global regularity of 2 dimensional minimal sets that are near a union of two planes, and prove that every global minimal set in $\mathbb{R}^4$ that looks like a union of two almost orthogonal planes at infinity is a cone. The main point is to use the topological properties of a minimal set at a large scale to control its behavior at smaller scales.

AMS classification. 28A75, 49Q10, 49Q20, 49K99

Key words. Minimal sets, Blow-in limit, Existence of singularities, Hausdorff measure, Elliptic systems.

1 Introduction

This paper deals with the local (resp. global) regularity of two-dimensional minimal sets in $\mathbb{R}^4$ that looks like the union of two almost orthogonal planes locally (resp. at infinity). The motivation is that we want to decide whether all global minimal sets in $\mathbb{R}^n$ are cones.

This Bernstein type of problem is of typical interest for all kinds of minimizing problems in geometric measure theory and calculus of variations. It is natural to ask how does a global minimizer look like, as soon as we know already the local regularity for minimizers. Well known examples are the global regularity for complete 2-dimensional minimal surfaces in $\mathbb{R}^3$, area or size minimizing currents in $\mathbb{R}^n$, or global minimizers for the Mumford-Shah functional. Some of them admit very good descriptions. See [2, 13, 12, 4] for further information.

Here our notion of minimality is defined in the setting of sets. Roughly speaking, we say that a set E is minimal when there is no deformation $F = \varphi(E)$, where φ is Lipschitz and $\varphi(x) - x$ is compactly supported, for which the Hausdorff measure $H^2(F)$ is smaller than $H^2(E)$. More precisely,

Definition 1.1 (Almgren competitor (Al competitor for short))**.** *Let E be a closed set in an open subset U of $\mathbb{R}^n$ and $d \leq n-1$ be an integer. An Almgren competitor for E is a closed set $F \subset U$ that can be written as $F = \varphi_1(E)$, where $\varphi_t : U \to U$ is a family of continuous mappings such that*

(1.2) $$\varphi_0(x) = x \textit{ for } x \in U;$$

(1.3) $$\text{the mapping } (t,x)\to\varphi_t(x) \text{ of } [0,1]\times U \text{ to } U \text{ is continuous;}$$

(1.4) $$\varphi_1 \text{ is Lipschitz,}$$

and if we set $W_t=\{x\in U\ ;\ \varphi_t(x)\neq x\}$ *and* $\widehat{W}=\bigcup_{t\in[0.1]}[W_t\cup\varphi_t(W_t)]$, *then*

(1.5) $$\widehat{W} \text{ is relatively compact in } U.$$

Such a φ_1 *is called a deformation in* U, *and* F *is also called a deformation of* E *in* U.

Definition 1.6 ((Almgren) minimal sets)**.** *Let* $0<d<n$ *be integers,* U *an open set of* $\mathbb{R}^n$. *A closed set* E *in* U *is said to be (Almgren) minimal of dimension* d *in* U *if*

(1.7) $$H^d(E\cap B)<\infty \text{ for every compact ball } B\subset U,$$

and

(1.8) $$H^d(E\backslash F)\leq H^d(F\backslash E)$$

for all Al competitors F *for* E.

This notion was introduced by Almgren to modernize Plateau's problem, which aims at understanding physical objects, such as soap films, that minimize the area while spanning a given boundary. The study of regularity and existence for these sets is one of the canonical interests in geometric measure theory.

Our goal is to show that every minimal set in $\mathbb{R}^n$ is a cone. The general idea is the following.

Let E be a $d-$dimensional reduced Almgren minimal set in $\mathbb{R}^n$. Reduced means that there is no unnecessary points. More precisely, we say that E is reduced when

(1.9) $$H^d(E\cap B(x,r))>0 \text{ for } x\in E \text{ and } r>0.$$

Recall that the definition of minimal sets is invariant modulo sets of measure zero, and it is not hard to see that for each Almgren (resp. topological) minimal set E, its closed support E^* (the reduced set $E^*\subset E$ with $H^2(E\backslash E^*)=0$) is a reduced Almgren (resp. topological) minimal set. Hence we can restrict ourselves to discussing only reduced minimal sets.

Now fix any $x\in E$, and set

(1.10) $$\theta_x(r)=r^{-d}H^d(E\cap B(x,r)).$$

This density function θ_x is nondecreasing for $r\in]0,\infty[$ (cf.[5] Proposition 5.16). In particular the two values

(1.11) $$\theta(x)=\lim_{t\to 0^+}\theta_x(t) \text{ and } \theta_\infty(x)=\lim_{t\to\infty}\theta_x(t)$$

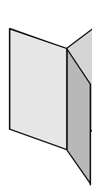
exist, and are called density of E at x, and density of E at infinity respectively. It is easy to see that $\theta_\infty(x)$ does not depend on x, hence we shall denote it by θ_∞.

Theorem 6.2 of [5] says that if E is a minimal set, $x\in E$, and $\theta_x(r)$ is a constant function of r, then E is a minimal cone centered on x. Thus by the monotonicity of the density functions $\theta_x(r)$ for any

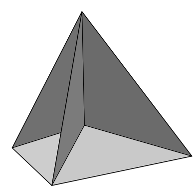
$x \in E$, if we can find a point $x \in E$ such that $\theta(x) = \theta_\infty$, then E is a cone and we are done.

On the other hand, the possible values for $\theta(x)$ and θ_∞ for any E and $x \in E$ are not arbitrary. By Proposition 7.31 of [5], for each x, $\theta(x)$ is equal to the density at the origin of a $d-$dimensional Al-minimal cone in $\mathbb{R}^n$. An argument around (18.33) of [5], which is similar to the proof of Proposition 7.31 of [5], gives that $\theta(x)$ is also equal to the density at the origin of a $d-$dimensional Al-minimal cone in $\mathbb{R}^n$. In other words, if we denote by $\Theta_{d,n}$ the set of all possible numbers that could be the density at the origin of a $d-$dimensional Almgren-minimal cone in $\mathbb{R}^n$, then $\theta_\infty \in \Theta_{d,n}$, and for any $x \in E$, $\theta(x) \in \Theta_{d,n}$.

Thus we restrict the range of θ_∞ and $\theta(x)$. Recall that the set $\Theta_{d,n}$ is possibly very small for any d and n. For example, $\Theta_{2,3}$ contains only three values: 1 (the density of a plane), 1.5 (the density of a $\mathbb{Y}$ set, which is the union of three closed half planes with a common boundary L, and that meet along the line L with 120° angles), and d_T (is the density of a $\mathbb{T}$ set, i.e., the cone over the 1-skeleton of a regular tetrahedron centered at 0). (See the figure below).



a $\mathbb{Y}$ set



a $\mathbb{T}$ set

Recall that the reason why θ_∞ has to lie in $\Theta_{d,n}$ is that, for any Al-minimal set E, all its blow-in limits have to be Al-minimal cones (cf. Argument around (18.33) of [5]). A blow-in limit of E is the limit of any converging (for the Hausdorff distance) subsequence of

$$E_r = r^{-1}E, r \to \infty. \tag{1.12}$$

Hence the value of θ_∞ implies that at sufficiently large scales, E looks like an Al-minimal cone of density θ_∞.

This is the same reason why $\theta(x) \in \Theta_{d,n}$. Here we look at the behavior of E_r when $r \to 0$, and the limit of any converging subsequence is called a blow-up limit (this might not be unique!). Such a limit is also an Al-minimal cone C (cf. [5] Proposition 7.31). This means, at some very small scales around each x, E looks like some Al-minimal cone C of density $\theta(x)$. In this case we call x a C type point of E.

After the discussion above, our problem will be solved if we can prove that every minimal cone C satisfies the following property:

$$\begin{aligned} &\text{There exists } \epsilon = \epsilon_C > 0, \text{ such that for every minimal set } E, \text{ if } d_{0,1}(C,E) < \epsilon, \text{ then} \\ &\text{there exists } x \in E \cap B(0,1) \text{ whose density } \theta(x) \text{ is the same as that of } C \text{ at the origin.} \end{aligned} \tag{1.13}$$

Here $d_{x,r}$ stands for the relative distance in the ball $B(x,r)$: for any closed sets E and F,

$$d_{x,r}(E,F)=\frac{1}{r}\max\{\sup\{d(y,F):y\in E\cap B(x,r)\},\sup\{d(y,E):y\in F\cap B(x,r)\}\}. \tag{1.14}$$

The discussion above uses only the values of densities at small scale and at infinity. A geometric intepretation is: there exists $x\in E\cap B(0,1)$ such that a blow-up limit C_x of E at x admits the same density as C at the origin.

So far we know that (1.13) is true for the planes and $\mathbb{Y}$ sets (see [5] Proposition 16.24). We do not know any minimal cone that does not verify the property (1.13). But there are at least two minimal cones for which we do not know whether (1.13) holds, either: the $\mathbb{T}$ set, and the sets $Y\times Y\in\mathbb{R}^4$, whose minimality has recently been proved in [11]. The topology of the set $Y\times Y$ is more complicated than that of $\mathbb{T}$ sets, and the situation of $\mathbb{T}$ sets is already tricky, see [10] for more detail.

In this paper we prove the property (1.13) for the unions of two almost orthogonal planes. Recall that in [9], we have proved the following

Theorem 1.15 (minimality of the union of two almost orthogonal planes, cf. [9] Thm 1.24)**.** *There exists $0<\theta_0<\frac{\pi}{2}$, such that if P^1 and P^2 are two planes in $\mathbb{R}^4$ whose characteristic angles (α_1,α_2) satisfy $\alpha_2\ge\alpha_1\ge\theta$, then their union $P^1\cup P^2$ is a minimal cone in $\mathbb{R}^4$.*

Here the characteristic angles describe the relative position between planes. Two planes P^1 and P^2 have characteristic angles (α_1,α_2) with $\alpha_2\ge\alpha_1\ge\theta$ means that there exists an orthonormal basis $\{e_i\}_{1\le i\le 4}$ of $\mathbb{R}^4$ such that P^1_α is generated by e_1 and e_2, and P^2_α is generated by $\cos\alpha_1 e_1+\sin\alpha_1 e_3$ and $\cos\alpha_2 e_2+\sin\alpha_2 e_4$. Each pair of $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\ge\alpha_1\ge\theta$ gives a minimal cone $P_\alpha=P^1\cup_\alpha P^2$, and the origin is called a singularity of type $\mathbb{P}_\alpha$ in the set P_α. These gives a continuous family of minimal cones with the same density at the origin, any two of which are not C^1 equivalent to each other. But still, we give them a general name, that is, each singularity of type $\mathbb{P}_\alpha$ is a singular point of type $2\mathbb{P}$.

So let us state our main results.

Theorem 1.16. *There is an angle $\theta_1\in[\theta_0,\frac{\pi}{2})$, (where θ_0 is the θ_0 in Theorem 1.15), and $\lambda>0$, such that for any $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\ge\alpha_1\ge\theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U\subset\mathbb{R}^4$, $B(x,r)\subset U$, and there is a reduced minimal cone P_α of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,r}(E,P_\alpha)\le\lambda$, then $E\cap B(x,r/100)$ contains (at least) a $2\mathbb{P}$ type point.*

A direct corollary to this is the expected global regularity for minimal sets that look like a union of two plane at the infinity:

Theorem 7.1. *Let θ_1 be as in Theorem 1.16. Then for any $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\ge\alpha_1\ge\theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $\mathbb{R}^4$ such that one blow-in limit of E at infinity is P_α (i.e., there exists a sequence of numbers $r_n\to\infty$, and the sequence of sets $r_n^{-1}(E)$ converge to P_α under the Hausdorff distance as $n\to\infty$), then E is a $\mathbb{P}_\alpha$ set.*

Besides the global regularity, the property (1.13) helps also to control the the relative distances $d_{x,r}$ between a minimal set and minimal cones in the balls $B(x,r)$ and the local speed of decay of the density function $\theta_x(r)$, because this property gives a lower bound of $\theta_x(r)$. When we prove (1.13) for a minimal cone C, we can get nicer local regularity results, that is, if a minimal set is very near C in a ball, then it should be equivalent to C in a smaller ball through a bi-Hölder homeomorphism (C^1 diffeomorphism in good cases). So here Theorem 1.16 has another useful corollary:

Theorem 7.2. *Let θ_1 be as in Theorem 1.16. Then there exists a $\epsilon>0$ such that for any $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\geq\alpha_1\geq\theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U\subset\mathbb{R}^4$, $B(x,100r)\subset U$, and there is a reduced minimal cone $P_\alpha+x$ of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,100r}(E,P_\alpha)\leq\epsilon$, then there exists a minimal cone $P_{\alpha'}$ of type $2\mathbb{P}$ such that there is a C^1 diffeomorphism $\Phi:B(x,2r)\to\Phi(B(x,2r))$, such that $|\Phi(y)-y|\leq 10^{-2}r$ for $y\in B(x,2r)$, and $E\cap B(x,r)=\Phi(P_{\alpha'})\cap B(x,r)$.*

The proof of Theorem 1.16 will keep us busy until the end of Section 6, but let us already try to explain how it goes.

First notice that Theorem 1.16 is invariant under translation with respect to x, and homogenous with respect to r, so we can only restrict to the case where $x=0$ and $r=1$.

Section 2 is devoted to giving some regularity properties for a minimal set E that is close to P_α, but does not contain any point of type $2\mathbb{P}$. In particular, we use a stopping time argument to find a critical region, outside of which everything goes fine, and inside of which things begin to go bad. Here "bad" means that the set begins to get far away from P_α. The main idea is to control the measure of E in the good region by finer estimates, since there we have good regularity properties; and for the bad region we only control its measure roughly by projections. Part of the argument will be similar to the proof of minimality of P_α.

Section 3 is quite short, where we sum up a little what happens, and give a competitor for E, using minimal graphs.

Section 4 and 5 are devoted to giving some estimates for minimal graphs, using some basic estimates for elliptic systems. This leads to some useful control on the measure of the competitor defined in Section 3.

In Section 6 we conclude, using harmonic extensions and projection properties of the competitor.

We discuss the global regularity and local C^1 regularity of minimal sets that are near a P_α cone in Section 7.

In this article, some of the results and arguments cited in [5] exist also in some other (earlier) references, e.g. [14]. But for simplify the article, the author will cite [5] systematically throughout this article.

Some useful notation

In all that follows, minimal set means Almgren minimal set;

$[a, b]$ is the line segment with end points a and b;

$[a, b)$ is the half line with initial point a and passing through b;

$B(x, r)$ is the open ball with radius r and centered on x;

$\overline{B}(x, r)$ is the closed ball with radius r and center x;

$\overrightarrow{ab}$ is the vector $b - a$;

H^d is the Hausdorff measure of dimension d ;

$d_H(E, F) = \max\{\sup\{d(y, F) : y \in E, \sup\{d(y, E) : y \in F\}\}$ is the Hausdorff distance between two sets E and F.

$d_{x,r}$: the relative distance with respect to the ball $B(x, r)$, is defined by

$$d_{x,r}(E, F) = \frac{1}{r}\max\{\sup\{d(y, F) : y \in E \cap B(x, r)\}, \sup\{d(y, E) : y \in F \cap B(x, r)\}\}.$$

2 A stopping time argument, and regularity and projection properties for minimal sets near P_α

In this section we use a stopping time argument to control some large scale behavior for minimal sets that near P_α. Let us first introduce some notation.

For each $\alpha = (\alpha_1, \alpha_2) \in [0, \frac{\pi}{2}]^2$ and $i = 1, 2$, denote by $P_\alpha = P_\alpha^1 \cup P_\alpha^2$, where P_α^1 and P_α^2 are two planes in $\mathbb{R}^4$ with characteristic angles (α_1, α_2) (this is equivalent to say that there exists an orthonormal basis $\{e_i\}_{1\le i\le 4}$ of $\mathbb{R}^4$ such that P_α^1 is generated by e_1 and e_2, and P_α^2 is generated by $\cos\alpha_1 e_1 + \sin\alpha_1 e_3$ and $\cos\alpha_2 e_2 + \sin\alpha_2 e_4$). Set

$$C_\alpha^i(x, r) = (p_\alpha^i)^{-1}(B(0, r) \cap P_\alpha^i) + x, \tag{2.1}$$

where p_α^i is the orthogonal projection on P_α^i, and

$$D_\alpha(x, r) = C_\alpha^1(x, r) \cap C_\alpha^2(x, r). \tag{2.2}$$

So $C_\alpha^i(x, r)$ is a cylinder and $D_\alpha(x, r)$ is the intersection of two cylinders. It is not hard to see that $D_\alpha(x, r) \supset B(x, r)$ and $D_\alpha(0, 1) \cap P_\alpha = B(0, 1) \cap P_\alpha$.

We say that two sets E, F are ϵr near each other in an open set U if

$$d_{r,U}(E, F) < \epsilon, \tag{2.3}$$

where

$$d_{r,U}(E, F) = \frac{1}{r}\max\{\sup\{d(y, F) : y \in E \cap U\}, \sup\{d(y, E) : y \in F \cap U\}\}. \tag{2.4}$$

We set also

$$\begin{aligned} d_{x,r}^\alpha(E, F) &= d_{r,D_\alpha(x,r)}(E, F) \\ &= \frac{1}{r}\max\{\sup\{d(y, F) : y \in E \cap D_\alpha(x, r)\}, \sup\{d(y, E) : y \in F \cap D_\alpha(x, r)\}\}. \end{aligned} \tag{2.5}$$

Remark 2.6. *We should be clear about the fact that*

$$d_{r,U}(E,F) \neq \frac{1}{r} d_H(E\cap U, F\cap U). \tag{2.7}$$

To see this, we can take $U = D_\alpha(x,r)$, and set $E_n = \partial D_\alpha(x, r+\frac{1}{n})$ and $F_n = \partial D_\alpha(x, r-\frac{1}{n})$. Then we have

$$d^\alpha_{x,r}(E_n, F_n) \to 0 \tag{2.8}$$

and

$$\frac{1}{r} d_H(E_n \cap D_\alpha(x,r), F_n \cap D_\alpha(x,r)) = \frac{1}{r} d_H(E_n \cap D_\alpha(x,r), \emptyset) = \infty. \tag{2.9}$$

So $d_{r,U}$ measures rather how the part of one set in the open set U could be approximated by the other set, and vice versa. However we always have

$$d^\alpha_{x,r}(E,F) \leq \frac{1}{r} d_H(E\cap D_\alpha(x,r), F\cap D_\alpha(x,r)). \tag{2.10}$$

Now we give the proposition below, obtained by a stopping time argument.

Proposition 2.11. *There exists $\epsilon_0 > 0$, such that for any $\epsilon < \epsilon_0$, and $\alpha > \frac{\pi}{3}$, if E is a closed reduced set which is minimal in $D_\alpha(0,1)$, $d^\alpha_{0,1}(E, P_\alpha) < \frac{\epsilon}{10}$, and E contains no $2\mathbb{P}$ point in $B(0, \frac{1}{100})$, then there exists $r_E \in]0, \frac{1}{2}[$ and $o_E \in B(0, 12\epsilon)$ such that E is $2\epsilon r_E$ near $P_\alpha + o_E$ in $D_\alpha(o_E, 2r_E(1-12\epsilon))$, but not ϵr_E near $P_\alpha + q$ in $D_\alpha(o_E, r_E)$ for any $q \in \mathbb{R}^4$.*

Remark 2.12. *We will also use the construction for information about intermediate scales in the proof.*

Proof of Proposition 2.11.

We fix any ϵ and $\alpha = (\alpha_1, \alpha_2) > \frac{\pi}{3}$, and set $s_i = 2^{-i}$ for $i \geq 0$. Set $D(x,r) = D_\alpha(x,r), d_{x,r} = d^\alpha_{x,r}$ for short.

We proceed in the following way.

Step 1: Denote by $q_0 = q_1 = O$, then in $D(q_0, s_0)$, E is ϵs_0 near $P_\alpha + q_1$ by hypothesis.

Step 2: If in $D(q_1, s_1)$, the set E is not ϵs_1 near $P_\alpha + q$ for any q, we stop; if not, there exists a q_2 such that E is ϵs_1 near $P_\alpha + q_2$ in $D(q_1, s_1)$. Here we also ask ϵ to be small enough (say, $\epsilon < \frac{1}{100}$) so that $q_2 \in D(q_1, \frac{1}{2}s_1)$, thanks to the conclusion of step 1. Then in $D(q_1, s_1)$, we have simultaneously :

$$d_{q_1,s_1}(E, P_\alpha + q_1) \leq s_1^{-1} d_{q_0,s_0}(E, P_\alpha + q_1) \leq 2\epsilon \ ; \ d_{q_1,s_1}(E, P_\alpha + q_2) \leq \epsilon. \tag{2.13}$$

Let us verify that (2.13) implies that $d_{q_1,\frac{1}{2}s_1}(P_\alpha + q_1, P_\alpha + q_2) \leq 12\epsilon$ when ϵ is small, say, $\epsilon < \frac{1}{100}$. In fact, for each $z \in D(q_1, \frac{1}{2}s_1) \cap (P_\alpha + q_1)$, we have $d(z, E) \leq d_{q_0,s_0}(E, P_\alpha + q_1) \leq \epsilon$, hence there exists $y \in E$ such that $d(z,y) \leq \epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + \epsilon) \subset D(q_1, s_1)$, and hence $d(y, P_\alpha + q_2) \leq s_1^{-1} d_{q_1,s_1}(E, P_\alpha + q_2) \leq 2\epsilon$, therefore $d(z, P_\alpha + q_2) \leq d(z,y) + d(y, P_\alpha + q_2) \leq 3\epsilon$.

On the other hand, suppose $z \in D(q_1, \frac{1}{2}s_1) \cap (P_\alpha + q_2)$, we have $d(z, E) \leq s_1^{-1} d_{q_1,s_1}(P_\alpha + q_2, E) \leq 2\epsilon$, hence there exists $y \in E$ such that $d(z,y) \leq 2\epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + 2\epsilon) \subset$

$D(q_0,s_0)$, and hence $d(y,P_\alpha+q_1)\leq d_{q_0,s_0}(E,P_\alpha+q_1)\leq\epsilon$, which implies $d(z,P_\alpha+q_1)\leq d(z,y)+d(y,P_\alpha+q_1)\leq 3\epsilon$.

As a result

$$d_{q_1,\frac{1}{2}s_1}(P_\alpha+q_1,P_\alpha+q_2)\leq(\frac{1}{2}s_1)^{-1}\times 3\epsilon=12\epsilon, \tag{2.14}$$

hence $d_{q_1,\frac{1}{2}s_1}(q_1,q_2)\leq 24\epsilon$, and therefore $d(q_1,q_2)\leq 6\epsilon=12\epsilon s_1$.

Now we define our iteration process (notice that it depends on ϵ, so we also call it a ϵ-process).

Suppose that all $\{q_i\}_{i\leq n}$ have already been defined, with

$$d(q_i,q_{i+1})\leq 12s_i\epsilon=12\times 2^{-i}\epsilon \tag{2.15}$$

for $0\leq i\leq n-1$, and hence

$$d(q_i,q_j)\leq 24\epsilon s_{\min(i,j)}=2^{-\min(i,j)}\times 24\epsilon \tag{2.16}$$

for $0\leq i,j\leq n$. Moreover, for all $i\leq n-1$, E is ϵs_i near $P_\alpha+q_{i+1}$ in $D(q_i,s_i)$. We say that the process does not stop at step n. In this case

Step n+1 : We look at the situation in $D(q_n,s_n)$.

If E is not ϵ near any $P_\alpha+q$ in this ball of radius s_n, we stop, since we have found the $o_k=q_n,r_k=s_n$ as desired. In fact, since $d(q_{n-1},q_n)\leq 12\epsilon s_{n-1}$, we have $D(q_n,2s_n(1-12\epsilon))=D(q_n,s_{n-1}(1-12\epsilon))\subset D(q_{n-1},s_{n-1})$, and hence

$$\begin{aligned}d_{q_n,2s_n(1-12\epsilon)}(P_\alpha+q_n,E)&\leq(1-12\epsilon)^{-1}d_{q_{n-1},s_{n-1}}(P_\alpha+q_n,E)\\&\leq\frac{\epsilon}{1-12\epsilon}.\end{aligned} \tag{2.17}$$

Moreover

$$d(o_k,O)=d(q_n,q_1)\leq 2^{-\min(1,n)}\times 24\epsilon=12\epsilon. \tag{2.18}$$

Otherwise, we can find a $q_{n+1}\in\mathbb{R}^4$ such that E is still ϵs_n near $P_\alpha+q_{n+1}$ in $D(q_n,s_n)$. Then since ϵ is small, $q_{n+1}\in D(q_n,\frac{1}{2}s_n)$. Moreover we have as before $d(q_{n+1},q_n)\leq 12\epsilon s_n$, and for $i\leq n-1$,

$$d(q_i,q_{n+1})\leq\sum_{j=i}^n d(q_j,q_{j+1})\leq\sum_{j=i}^n 12\times 2^{-j}\epsilon\leq 2^{-j}\times 24\epsilon=2^{-\min(i,n+1)}\times 24\epsilon. \tag{2.19}$$

Thus we have obtained our q_{n+1}.

Now all we have to do is to prove that for every ϵ small enough, this process has to stop at a finite step. For this purpose we need the following proposition.

Proposition 2.20. *There exists $\theta_1'\in[\theta_0,\frac{\pi}{2})$, and for any $l\in]0,\frac{1}{2}]$, there exists $\epsilon_l\in]0,\frac{1}{2}[$, such that for any $\alpha>\theta_1'$, $\epsilon\leq\epsilon_l$, and E as in Proposition 2.11, if the $\epsilon-$process does not stop before the step n, then*

(1) The part $E\cap(D_\alpha(0,\frac{39}{40})\backslash D_\alpha(q_n,\frac{1}{10}s_n))$ is composed of two disjoint pieces $G^i,i=1,2$, such that:

$$G^i \text{ is the graph of a } C^1 \text{ map } g^i:C_\alpha^i(0,\frac{39}{40})\backslash C_\alpha^i(q_n,\frac{1}{10}s_n)\cap P_\alpha^i\to {P_\alpha^i}^\perp \tag{2.21}$$

with

$$\|\nabla g^i\|_\infty < l \leq \frac{1}{2}; \tag{2.22}$$

(2) For every $\frac{1}{10}s_n \leq t \leq s_n$

$$E \cap (D_\alpha(0,1) \backslash D_\alpha(q_n,t)) = G_t^1 \cup G_t^2 \tag{2.23}$$

where G_t^1, G_t^2 *do not meet each other. Moreover*

$$P_\alpha^i \cap (D_\alpha(0,1) \backslash C_\alpha^i(q_n,t)) \subset p_\alpha^i(G_t^i) \tag{2.24}$$

where p_α^i *is the orthogonal projection on* $P_\alpha^i, i = 1,2$*;*

Remark 2.25. *If we take the optimal* ϵ_l *for each* l *such that Proposition 2.20 holds, then obviously for any* $l \leq l'$, $\epsilon_l \leq \epsilon_{l'}$.

We will not prove this proposition, see [9] Proposition 6.1 (1) (2) for the proof. But we'll use it to finish our Proposition 2.11.

Remark 2.26. *In fact we need all the properties stated in [9] Proposition 6.1 for our set* E*. For (1) and (2) in [9] Proposition 6.1, the arguments there can be applied directly here to our set* E *with no change. But for (3) and (4), the proof in [9] Proposition 6.1 uses some special property of* E_k*, which are not necessarily true for our set* E *here. Hence we will treat the property of surjective projections ((4) of [9] Proposition 6.1) later in a different way.*

So let ϵ_0 be the $\epsilon_{\frac{1}{2}}$ in Proposition 2.20. Suppose that the $\epsilon-$process does not stop at any finite step, and we'll try to get a contradiction. By (1) of Proposition 2.20, for any n, $E \cap (D_\alpha(0,1) \backslash D_\alpha(q_n, \frac{1}{10}s_n))$ is composed of two disjoint graphs G^i on $[C_\alpha^i(0,1) \backslash C_\alpha^i(q_n, \frac{1}{10}s_n)] \cap P_\alpha^i, i = 1,2$. Denote by $\Delta_n = D_\alpha(q_n, s_n)$.

Notice that by (2.19), with $\epsilon < \frac{1}{100}$, the sets $\Delta_n = D_\alpha(q_n, s_n)$ are in fact a sequence of non degenerate compact balls, with

$$\Delta_n \subset \Delta_{n-1}, n \in \mathbb{N}, \lim_{n\to\infty} \mathrm{diam}(\Delta_n) \to 0, \tag{2.27}$$

Hence there exists a point $p \in B(0, \frac{1}{2})$, such that $\{p\} = \cap_n \Delta_n$. Then p is also the limit of q_n, hence it lies in $B(0, \frac{1}{100})$. By (1) of Proposition 2.20, for any $r \in (0, \frac{1}{2})$, $E \cap D(p, \frac{1}{2}) \backslash D(p,r)$ is composed of the union of two disjoint graphs on $P_\alpha^i \cap C_\alpha^i(p, \frac{1}{2}) \backslash C_\alpha^i(p,r)$. As a result , $E \cap D(p, \frac{1}{2}) \backslash \{p\}$ is composed of two C^1 graphs on $P_\alpha^i \cap C_\alpha^i(p, \frac{1}{2}) \backslash \{p\}$. Denote by G^i these two graphs. By (2.22), they are both $\frac{1}{2}$-Lipschitz. Now E is closed hence $p \in E$. Then for each $i = 1,2$, $G^i \cup \{p\}$ is a $\frac{1}{2}$-Lipschitz graph on $P_\alpha^i \cap C_\alpha^i(p, \frac{1}{2})$, and hence $E \cap D_\alpha(p, \frac{1}{2})$ is composed of the disjoint union of these two $\frac{1}{2}$-Lipschitz graphs. Now we define $\varphi : E \cap D_\alpha(p, \frac{1}{2}) \to P_\alpha + p$, where the restriction of φ to each $G^i \cup \{p\}$ is just the orthogonal projection to $P_\alpha^i + p$. Then it is easy to check that φ is a Lipschitz homeomorphism. That is, E is bi-Lipschitz homeomorphic to P_α in $D_\alpha(p, \frac{1}{2})$.

We want to prove that p is a point of type $2\mathbb{P}$. Take any blow-up limit C of E at the point p. Then C is a minimal cone. By the bi-Hölder regularity for 2-dimensional minimal sets, near the point p, E

is locally bi-Hölder equivalent to C. But E is also bi-Lipschitz equivalent to p_α near p, hence the two minimal cones P_α and C are topologically the same. As a consequence, $P_\alpha\cap\partial B(0,1)$ and $C\cap\partial B(0,1)$ are topologically the same, therefore, $C\cap\partial B(0,1)$ is the union of two topological circles. But by the description of 2-dimensional minimal cones (cf.[5], Proposition 14.1), the intersection of any minimal cone with the unit sphere is a finite union of great circles and arcs of great circles that meet at their extremities by group of three with $120°$ angles. Here in our case, we can deduce that $C\cap\partial B(0,1)$ is the union of two circles. Hence C is a minimal cone of type $2\mathbb{P}$.

Hence the point p is a point of type $2\mathbb{P}$. This contradicts the fact that $E\cap B(0,\frac{1}{100})$ contains no point of type $2\mathbb{P}$, because $p\in B(0,\frac{1}{100})$.

Thus we complete the proof of Proposition 2.11. $\square$

Next we still have to prove some property of surjective projection, as remarked in Remark 2.26.

Proposition 2.28. *Take $\epsilon\leq\epsilon_0$, and take α and E as in Proposition 2.20. Then for any $n\geq 1$, if the $\epsilon-$process does not stop before the step n, then the orthogonal projections $p_\alpha^i: E\cap\overline{D}_\alpha(q_n,t)\to P_\alpha^i\cap\overline{C}_\alpha^i(q_n,t), i=1,2$ are surjective, for all $\frac{1}{9}s_n\leq t\leq s_n$.*

Proof. Fix a such n. Set $s_i=2^{-i}$ for $i\geq 0$. Set $D(x,r)=D_\alpha(x,r), C^i(x,r)=C_\alpha^i(x,r), d_{x,r}=d_{x,r}^\alpha$ for short. By (1) of Proposition 2.20, the part $E\cap(D_\alpha(0,\frac{39}{40})\backslash D_\alpha(q_n,\frac{1}{10}s_n))$ is composed of two disjoint pieces $G^i, i=1,2$, such that:

$$G^i \text{ is the graph of a } C^1 \text{ map } \ g^i: C_\alpha^i(0,\frac{39}{40})\backslash C_\alpha^i(q_n,\frac{1}{10}s_n)\cap P_\alpha^i\to {P_\alpha^i}^\perp \tag{2.29}$$

with

$$||\nabla g^i||_\infty<\frac{1}{2}. \tag{2.30}$$

Thus $G^i\cap\partial C^i(0,\frac{39}{40})$ is a nice C^1 curve, which is the graph of g^i on $P_\alpha^i\cap\partial C^i(0,\frac{39}{40})$, and g^i is $\frac{1}{2}$-Lipschitz. Denote by $\gamma^i=g^i|_{P_\alpha^i\cap\partial C^i(0,\frac{39}{40})}$. Then $||\gamma^i||_\infty\leq\frac{\epsilon}{10}$ by hypothesis.

Now we define a set Q as follows. First, $Q\subset\overline{B}(0,1)$, and $Q\backslash D(0,\frac{39}{40})=E\backslash D(0,\frac{39}{40})$. Inside $D(0,\frac{3}{4})$, $Q\cap\overline{D}(0,\frac{3}{4})=P_\alpha\cap\overline{D}(0,\frac{3}{4})$, the union of two planes. For the part on the annulus $D(0,\frac{39}{40})\backslash\overline{D}(0,\frac{3}{4})$, we just use two graphs of affine functions to join $P_\alpha^i\cap\partial D(0,\frac{3}{4})$ and γ^i. That is, we define $h^i: P_\alpha^i\cap D(0,\frac{39}{40})\backslash\overline{D}(0,\frac{3}{4})\to {P_\alpha^i}^\perp$, for any $x\in P_\alpha^i\cap D(0,\frac{39}{40})\backslash\overline{D}(0,\frac{3}{4})(\frac{3}{4},\frac{39}{40})$, $h^i(x)=\frac{|x|-\frac{3}{4}}{\frac{39}{40}-\frac{3}{4}}\gamma^i(\frac{39x}{40|x|})$.

Thus for any $x\in D(0,\frac{39}{40})\backslash\overline{D}(0,\frac{3}{4})$, $|\frac{\partial}{\partial r}h^i(x)|=\frac{1}{\frac{39}{40}-\frac{3}{4}}|\gamma^i(\frac{39x}{40|x|})|\leq\frac{40}{9}\frac{\epsilon}{100}\leq\frac{\epsilon}{20}\leq\frac{1}{2000}$, and $|\frac{\partial}{\partial\theta}(x)|\leq Lip(\gamma^i)\leq\frac{1}{2}$, hence the tangent direction derivative is less than

$$\frac{1}{|x|}|\frac{\partial}{\partial\theta}(x)|\leq\frac{1}{2}/\frac{3}{4}=\frac{2}{3}. \tag{2.31}$$

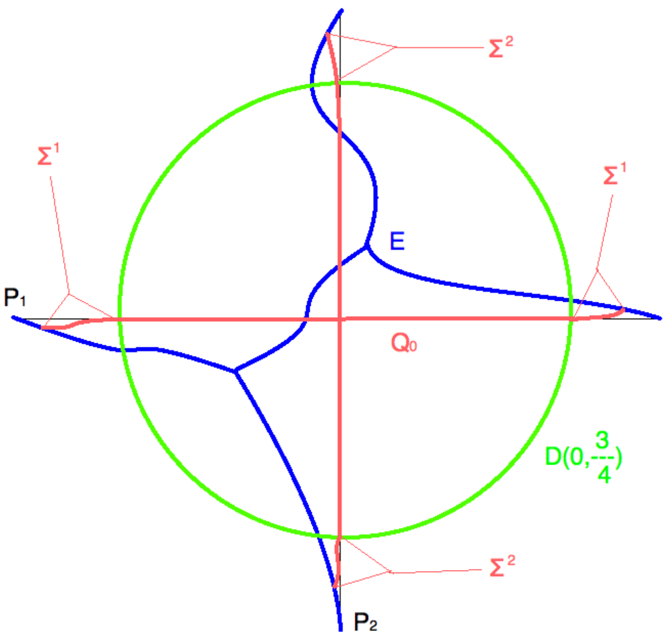
Hence we have

$$\text{Lip } h^i\leq\max\{\frac{1}{2000},\frac{2}{3}\}=\frac{2}{3}. \tag{2.32}$$

Thus the map $H^i: P_\alpha^i\cap D(0,\frac{39}{40})\backslash\overline{D}(0,\frac{3}{4})\to\mathbb{R}^4: x\mapsto(x,h^i(x))$ is $(1+(\frac{2}{3})^2)^{\frac{1}{2}}=\frac{\sqrt{13}}{3}$-Lipschitz. So if

we denote by Σ^i the graph of h^i, then

$$H^2(\Sigma^i) = H^2(H^i(P^i_\alpha \cap D(0, \frac{39}{40})\backslash \overline{D}(0, \frac{3}{4})) \leq (\frac{\sqrt{13}}{3})^2) H^2(P^i_\alpha \cap D(0, \frac{39}{40})\backslash \overline{D}(0, \frac{3}{4})) \tag{2.33}$$
$$= \frac{897}{1600}\pi \leq \frac{9\pi}{16}, i = 1, 2.$$



2-1

Let $Q = [E\backslash D(0, \frac{39}{40})] \cup \Sigma^1 \cup \Sigma^2 \cup [P_\alpha \cap D(0, \frac{3}{4})]$, and $Q_0 = Q \cap D(0, \frac{39}{40})$. (See Figure 2-1.) Set $Q^i = \Sigma^i \cup [P^i_\alpha \cap D(0, \frac{3}{4})]$, then Q_0 is the almost disjoint union $Q^1 \cup Q^2$. For each $i = 1, 2$,

$$H^2(Q^i) = H^2(\Sigma^i) + H^2(P^i_\alpha \cap D(0, \frac{3}{4})) \leq \frac{9\pi}{16} + \frac{9\pi}{16} = \frac{9\pi}{8}. \tag{2.34}$$

Notice that the set Q_0 is a C^1 version of $P_\alpha \cap D(0, \frac{3}{4})$, and $Q^i, i = 1, 2$ are its two flat parts as P^i_α.

Now suppose that for some $t \in [\frac{1}{9}s_n, s_n)$, for example the projection $p^1_\alpha : E \cap D(q_n, t) \to P^1_\alpha \cap C^1(q_n, t)$ is not surjective. Then we are going to prove that we can deform E to $[Q\backslash Q_0] \cup Q^2$, and deduce a contradiction.

So take a point $p \in P^1_\alpha \cap \overline{C}^1(q_n, t)$ which does not admit a pre-image in $E \cap \overline{D}(q_n, t)$. Since the set $E_t := E \cap \overline{D}(q_n, t)$ is compact, its projection $p^1_\alpha(E_t)$ is also compact, which means that we can pick $p \in P^1_\alpha \cap C^1(q_n, t)\backslash p^1_\alpha(E_t)$ and $r \in (0, \frac{t}{10})$ such that $B(p, r) \cap P^1_\alpha \subset P^1_\alpha \cap C^1(q_n, t)\backslash p^1_\alpha(E_t)$, and moreover $0 \notin B(p, 3r)$.

Now the set $E_t \subset \overline{D}(q_n, t)\backslash {p^1_\alpha}^{-1}(B(p, r) \cap P^1_\alpha)$. Take an orthogonal union of two planes $P_0 = P^1_0 \cup_\perp P^2_0$ in $\mathbb{R}^4$, denote by p^i_0 the orthogonal projection on $P^i_0, k = 1, 2$, take a point $p_0 \in P^1_0$ such that $d(p_0, o) = \frac{1}{2}$.

Then we can easily find a Bi-Lipschitz mapping $\varphi : \overline{D}(q_n, t)\backslash {p^1_\alpha}^{-1}(B(p, r) \cap P^1_\alpha) \to \overline{D}(0, 1)\backslash {p^1_0}^{-1}(B(p_0, \frac{1}{4}) \cap P^1_0)$, such that $\varphi(E_t \cap D(q_n, t)\backslash D(q_n, \frac{1}{10}s_n)) = P_0 \cap D(0, 1)\backslash D(0, \frac{3}{4})$ (because in the annulus $D(q_n, t)\backslash D(q_n, \frac{1}{10}s_n)$,

the set E is still a C^1 graph of P_α).

For any point $x \in D(0,1)$, write $x=(x_1,x_2)$, where $x_i = p_0^i(x) \in B^i(0,1), i=1,2$ ($B^i(0,1)$ is the unit ball of the plane P_0^i). We define $\psi : D(0,1)\backslash {p_0^1}^{-1}(B((p_0,\frac{1}{4})\cap P_0^1) \to D(0,1)\cap P_0\backslash {p_0^1}^{-1}(B((p_0,\frac{1}{4})\cap P_0^1)$ as follows:

$$\psi(x) = \begin{cases} p_0^1(x), & x_2 < \frac{3}{4}; \\ (x_1, 4x_2-3), & x_2 \geq \frac{3}{4}. \end{cases} \tag{2.35}$$

Then ψ is a Lipschitz map, which maps $[C^1(0,1)\cap C^2(0,\frac{3}{4})]\cup[P_0\cap D(0,1)]$ to $P_0\cap D(0,1)$, and $\psi|_{P_0\cap\partial D(0,1)} = Id$. In particular, $\psi(\varphi(E_t)) \subset P_0\cap D(0,1)\backslash {p_0^1}^{-1}(B(p_0,\frac{1}{4})\cap P_0^1)$.

Thus the map $f_1 = \varphi^{-1}\circ\psi\circ\varphi$ maps E_t to $P_\alpha \cap D(q_n,t)\backslash D(q_n,\frac{1}{10}s_n)$, and $f_1|_{E\cap\partial D(q_n,t)} = id$.

We can extend f_1 to a Lipschitz map from $D(0,\frac{39}{40}) \to D(0,\frac{39}{40})$, such that $f_1|_{E\cap D(0,\frac{39}{40})\backslash D(q_n,t)} = id$ and $f_1|_{D(0,\frac{39}{40})\backslash D(0,\frac{1}{2})} = id$.

Then f_1 is a deformation of E in $D(0,\frac{39}{40})$, which sends $E\cap D(0,\frac{39}{40})$ to $Q_0\backslash[B(p,r)\cap P_\alpha^1]$, this is the union of Q^2 and Q^1 minus a hole $B(p,r)\cap P_\alpha^1$. So we can keep on the deformation, and take the map f_2 which deforms $Q^1\backslash[B(p,r)\cap P_\alpha^1]$ to a set $E^1 = \{0\}\cup\partial Q^1\cup C$ of measure zero, where C is a segment that connects the origin and ∂Q^1 and keeps Q^2 fixed. Then the map $f = f_2\circ f_1$ sends $E\backslash D(0,\frac{39}{40})$ to $Q^2\cup E^1$, hence the measure

$$H^2(E\cap D(0,\frac{39}{40})) = H^2(Q^2) \leq \frac{9\pi}{8}. \tag{2.36}$$

The map f is Lipschitz, and its restriction to $Q_0\cap\partial D(0,\frac{39}{40})$ is the identity. We extend f to a Lipschitz map on $D(0,1)$, still denoted by f, such that $f = id$ near the boundary of $D(0,1)$. Thus by the minimality of E, and since f does not move $E\backslash D(0,\frac{39}{40})$, we have

$$H^2(E\cap D(0,\frac{39}{40})) \leq H^2(f(E\cap D(0,\frac{39}{40})) \leq \frac{9\pi}{8}. \tag{2.37}$$

However since $n>1$, we have $s_n < \frac{1}{2}$. By (1) of Proposition 2.20, we have

$$\begin{aligned} H^2(E\cap D(0,\frac{39}{40})) &\geq H^2(G^1)+H^2(G^2) \geq H^2(p_\alpha^1(G^1)) + H^2(p_\alpha^2(G^2)) \\ &= \sum_{i=1,2} H^2(P_\alpha^i\cap C^i(0,\frac{39}{40})\backslash C^i(q_n,\frac{1}{10}s_n)) \\ &\geq \sum_{i=1,2} H^2(P_\alpha^i\cap C^i(0,\frac{39}{40})\backslash C^i(q_n,\frac{1}{20})) \\ &= 2\times\pi((\frac{39}{40})^2-(\frac{1}{20})^2) = \frac{1517}{800}\pi > \frac{9\pi}{8}, \end{aligned} \tag{2.38}$$

which leads to a contradiction.

This completes the proof of Proposition 2.28. □

3 A competitor, and estimates for minimal graphs

Let θ_1', α be as in Proposition 2.20, let $\epsilon = \epsilon_0$, μ be chosen later, and let E be as in Proposition 2.11, that is, $d_{0,1}^\alpha < \frac{\epsilon}{10}$, and E contains no $2\mathbb{P}$ type point in $B(0, \frac{1}{100})$. We want to construct a competitor for E, and show that if $d_{0,1}^\alpha$ is sufficiently small, this competitor admits necessarily less measure than E, and thus leads to a contradiction.

Let us point out that the condition $d_{0,1}^\alpha < \frac{\epsilon}{10}$ is a general qualitative one, which guarantees that E satisfies the regularity properties in Proposition 2.20 and 2.28. To make the necessary finer estimates for measures of E and its competitor, we still have to get the "λ-near" condition as in Theorem 1.16.

So by Proposition 2.11, there is a $r_E \in]0, \frac{1}{2}[$, $o_E \in B(0, \frac{1}{2}\epsilon_0)$ such that the conclusion in Proposition 2.11 holds for E. Denote by $\gamma^i : \partial B(0, \frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$ the C^1 curve $g^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i}$. Suppose that $||\gamma^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i}||_{C^1} \leq \mu$.

The idea of the construction of the competitor is not complicated. We take, for each i, a minimal graph Σ^i which is the graph of a function $f^i : B(0, \frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$ such that $f^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i} = \gamma^i$. Take $\Sigma = \Sigma^1 \cup \Sigma^2$. Then hopefully when μ is small enough, these two graphs are very flat at the center, so that Σ is very similar to P_α. Thus we can deform $E \cap D_\alpha(0, \frac{1}{2})$ to a subset of Σ in a Lipschitz manner, while keeping $E \cap \partial D_\alpha(0, \frac{1}{2})$ unchanged. Hence Σ contains a competitor of E in $D_\alpha(0, \frac{1}{2})$. By the minimality of E, the measure of Σ has to be larger than that of $E \cap D_\alpha(0, 1)$. But we are going to show that when μ is small enough, this is not true.

Before we go down to the following two sections, which will be devoted to giving some estimates for minimal graphs, let us already explain what happens.

We want to compare the measures of $E \cap D_\alpha(0, \frac{1}{2})$ and Σ. Outside $D(o_E, \frac{1}{10}r_E)$, by Proposition 2.20, E is also composed of two C^1 graphs G^i on the two annuli $P_\alpha^i \cap B(0, \frac{1}{2}) \backslash C^i((o_E, \frac{1}{10}r_E)$. So in this part, our goal is to compare the surface measure of Σ^i and G^i, that is, the graph of f^i and g^i. Notice that f^i and g^i coincide on $\partial P_\alpha^i \cap \partial B(0, \frac{1}{2})$, and on $P_\alpha^i \cap \partial B(o_E, \frac{1}{10}r_E)$, g^i is supposed to be ϵ-far from any plane, while f^i is almost a plane (this is the main result of Section 4). Then Section 5 is devoted to estimating the difference between these two graphs.

So this will help estimate the difference between measures of E and Σ on the annulus region $D_\alpha(0, \frac{1}{2}) \backslash D(o_E, \frac{1}{10}r_E)$. For the part of $E \cap D(o_E, \frac{1}{10}r_E)$, we estimate its measure by using projections.

4 Existence and estimates for derivatives for minimal graphs

Denote by $B = B(0, 1) \cap \mathbb{R}^2$ the unit disc in $\mathbb{R}^2$. Let γ be a C^1 function from ∂B to $\mathbb{R}^2$. Now by Theorems 4.1 and 4.2 of [8], there exists a function $f : \overline{B} \to \mathbb{R}^2$, whose graph $\Sigma_f = \{(x, f(x)) : x \in \overline{B}\} \subset \mathbb{R}^4$ is a minimal surface, $f|_{\partial B} = \gamma$, and $f \in C^0(\overline{B}) \cap C^\infty(B)$. In particular, by (c) of Theorem 4.1 of [8] and

the maximum principle for harmonic maps, we have

$$||f||_\infty \leq ||\gamma||_{L^\infty(\partial B)}. \tag{4.1}$$

Now suppose that $\mu = \max\{||\gamma||_{L^\infty(\partial B)}, ||D\gamma||_{L^\infty(\partial B)}\}$ is small, then by (4.1), $||f||_\infty \leq \mu$ is small. We want to prove that $|\nabla f|, |\nabla^2 f|, |\nabla^3 f|$ are also small in a neighborhood of 0, and are controlled by μ. More precisely, we state the following proposition.

Proposition 4.2. *There exists $\mu_0 > 0$, such that for any $\mu < \mu_0$, there exists a constant $C(\mu)$, with $\lim_{\mu\to 0} C(\mu) = 0$, such that if f is a minimal graph on $B(0,1)$, with*

$$\max\{||f|_{\partial B(0,1)}||_\infty, ||Df|_{\partial B(0,1)}||_\infty\} \leq \mu, \tag{4.3}$$

then

$$\max_{0\leq i\leq 3} ||\nabla^i f||_{L^\infty(B(0,\frac{3}{4}))} \leq C(\mu). \tag{4.4}$$

Proof.

First let us apply a regularity theorem on varifolds to get the initial estimate for ∇f, and then we can go into the machine of estimates for elliptic systems. Before stating the theorem, we give some useful notations below.

$G(n,d)$ denotes the Grassmann manifold $G(\mathbb{R}^n, d)$;

for every $T \in G(n,d)$, we denote by π_T the orthogonal projection on the d-plane represented by T;

for every measure ν on $\mathbb{R}^n$, $\theta^d(\nu, x) = \lim_{r\to 0} \frac{\nu B(a,r)}{\alpha(d) r^d}$ (if the limit exists) is the density of ν on x, where $\alpha(d)$ denotes the volume of the d-dimensional unit ball;

$\mathbb{V}_d(\mathbb{R}^n)$ denotes the set of all $d-$varifold in $\mathbb{R}^n$, i.e. all Radon measures on $G_d(\mathbb{R}^n) = \mathbb{R}^n \times G(n,d)$;

for each $V \in \mathbb{V}_d(\mathbb{R}^n)$, $||V||$ is the Radon measure on $\mathbb{R}^n$ such that for each $A \subset \mathbb{R}^n$, $||V||(A) = V(G_d(\mathbb{R}^n) \cap \{(x,S) : x \in A\})$;

$\delta(V)$ denotes the first variation of V, that is, the linear map from $\mathfrak{X}(R^n)$ to $\mathbb{R}$, defined by

$$\delta V(g) = \int Dg(x) \cdot \pi_S dV(x,S) \tag{4.5}$$

for $g \in \mathfrak{X}(\mathbb{R}^n)$. Here $\mathfrak{X}(\mathbb{R}^n)$ is the vector space of all C^∞ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ with compact support.

In our case, we are only interested in rectifiable varifolds. In fact, with each $d-$rectifiable set E we associate a $d-$varifold, denoted by V_E, in the following sense: for each $B \subset \mathbb{R}^n \times G(n,d)$, we have

$$V_E(B) = H^d\{x : (x, T_x E) \in B\}. \tag{4.6}$$

Recall that $T_x E$ is the d-dimensional tangent plane of E at x; it exists for almost all $x \in E$, because E is $d-$rectifiable. Then $||V_E|| = H^d|_E$. Moreover, the density $\theta^d(||V_E||, x)$ exists for almost all $x \in E$.

Theorem 4.7 (cf.[1] Regularity theorem at the beginning of section 8)**.** *Suppose $2 \leq d < p < \infty$, $q = \frac{p}{p-1}$. Corresponding to every $\epsilon \in]0,1[$ there is $\eta > 0$ with the following property:*

Suppose $0 < R < \infty$, $0 < \lambda < \infty$, $V \in \mathbb{V}_d(\mathbb{R}^n)$, $a \in spt||V||$ and

1) $\theta^d(||V||, x) \geq \lambda$ *for* $||V||$ *almost all* $x \in B(a, R)$*;*

2) $||V||B(a, R) \leq (1+\eta)\lambda\alpha(d)R^d$*;*

3) $\delta V(g) \leq \eta\lambda^{\frac{1}{p}} R^{\frac{d}{p-1}} (\int |g|^q \lambda ||V||)^{\frac{1}{q}}$ *whenever* $g \in \mathfrak{X}(\mathbb{R}^n)$ *and spt* $g \subset B(a, R)$*.*

Then there are $T \in G(n, d)$ *and a continuously differentiable function* $F : T \to \mathbb{R}^n$*, such that* $\pi_T \circ F = 1_T$*,*

$$||DF(y) - DF(z)|| \leq \epsilon(|y-z|/R)^{1-\frac{d}{p}} \text{ whenever } y, z \in T, \tag{4.8}$$

and

$$B(a, (1-\epsilon)R) \cap spt||V|| = B(a, (1-\epsilon)R) \cap \text{ image } F. \tag{4.9}$$

Remark 4.10. *1) In the theorem, since* $\pi_T \circ F = 1_T$*, we can see that* F *is in fact the graph of a* C^1 *function* f*, defined by* $f(t) = \pi_{T^\perp} F(t)$*, with* $t \in T$*,* $\pi_{T^\perp}$ *the orthogonal projection on the orthogonal space* $T^\perp$ *of* T*. Moreover* $||Df(t)|| \leq ||DF(t)||$ *for all* $t \in T$*.*

2) If E *is a minimal surface, then* V_E *is stationary, i.e.* $\delta V_E = 0$*. Hence the condition 3) is automatically true. In fact if we set* $g_t(x) = (1-t)x + tg(x)$*, then*

$$\delta V_E(g) = \frac{d}{dt} H^d(g_t(E \cap sptg)), \tag{4.11}$$

which can be deduced from the area formula. Thus if E *is a minimal surface,* $\delta V_E = 0$*.*

Now we want to apply Theorem 4.7 to our set Σ_f, so we have to check all the conditions in the theorem. We take $\lambda = 1, a = (0, f(0)), R = 1$, then 1) is true, by the fact that Σ_f is a C^∞ manifold; 3) is true by the Remark 4.10 2); for 2), notice first of all that $B(a, R) \cap \Sigma_f \subset \Sigma_f$, so we just have to estimate the surface of Σ_f. Notice that Lip $\gamma \leq \mu$, hence for the length of the graph of γ, denoted also by γ, we have

$$|\gamma| = \int_{\partial B} \sqrt{1 + |D\gamma|^2} \leq \int_{\partial B} \sqrt{1 + \mu^2} = 2\pi(1 + \mu^2). \tag{4.12}$$

Now by the isoperimetric inequality for minimal surface (cf. [3]), we have

$$4\pi H^2(\Sigma_f) \leq |\gamma|^2 = [2\pi(1+\mu^2)]^2, \tag{4.13}$$

which means

$$H^2(\Sigma_f \cap B(a, R)) \leq H^2(\Sigma_f) \leq (1+\mu^2)^2\pi. \tag{4.14}$$

Hence we can take μ small enough such that 2) holds for some η, such that (4.8) and (4.9) are true for some ϵ small, which give us that

$$||f||_{C^{1,\sigma}(B(0, \frac{8}{9}))} \leq C_1(\mu), \tag{4.15}$$

with $\lim_{\mu \to 0} C_1(\mu) = 0$.

Remark 4.16. *We might be able to use only the estimates for elliptic system to get this initial estimate, without using the powerful Theorem 4.7.*

For estimating higher order regularity of f, we have to refer to the minimal surface equation system and put everything in the machine of elliptic system.

First we give some notations.

Denote by $M_2(\mathbb{R})$ the set of 2×2 matrices on $\mathbb{R}$. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, denote by $|\begin{pmatrix} a & b \\ c & d \end{pmatrix}| = a^2+b^2+c^2+d^2$, and for any $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in M_2(\mathbb{R})$, define $< \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} >= aa'+bb'+cc'+dd'$. Denote by $\cdot$ the multiplication of matrices. Set, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \in M_2(\mathbb{R})$.

For any domain $\Omega \subset \mathbb{R}^2$, for any differentiable function $h : \Omega \to \mathbb{R}$, denote by h_x, h_y its two partial derivates. For any C^2 function $h = (h^1, h^2) : \mathbb{R}^2 \to \mathbb{R}^2$, with $h^i : \mathbb{R}^2 \to \mathbb{R}$ two C^1 functions, denote by ∇h the matrix valued function $\begin{pmatrix} h^1_x & h^2_x \\ h^1_y & h^2_y \end{pmatrix}$. And for any matrix valued function $f = \begin{pmatrix} f^1 & f^2 \\ f^3 & f^4 \end{pmatrix}$ on $\mathbb{R}^2$, we define $div f = (f^1_x + f^3_y, f^2_x + f^4_y) \in \mathbb{R}^2$.

Then we have

$$H^2(\Sigma_h) = \int_\Omega \sqrt{1+|\nabla h|^2 + (\det \nabla h)^2}. \tag{4.17}$$

Denote by $S(h) = |\nabla h|^2 + (\det \nabla h)^2$ for any h.

Σ_f is a minimal submanifold, hence it is stable with respect to any local perturbation. More precisely, for any C^∞ function $\varphi : \overline{B} \to \mathbb{R}^2$ with $\varphi|_{\partial B} = 0_{\mathbb{R}^2}$, we have

$$\frac{d}{dt}|_{t=0} H^2(\Sigma_{f+t\varphi}) = 0. \tag{4.18}$$

(4.17) and (4.18) gives that, for any C^∞ function $\varphi : \overline{B} \to \mathbb{R}^2$ with $\varphi|_{\partial B} = 0_{\mathbb{R}^2}$,

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \int_B \sqrt{1+|\nabla(f+t\varphi)|^2 + (\det \nabla(f+t\varphi))^2} \\ &= \int_B \frac{d}{dt}|_{t=0} \sqrt{1+|\nabla(f+t\varphi)|^2 + (\det \nabla(f+t\varphi))^2} \\ &= \int_B \frac{\frac{d}{dt}|_{t=0} < \nabla(f+t\varphi), \nabla(f+t\varphi) > + \frac{d}{dt}|_{t=0} (\det \nabla(f+t\varphi))^2}{2\sqrt{1+S(f)}} \\ &= \int_B \frac{< \nabla f, \nabla\varphi > + \det(\nabla f) \frac{d}{dt}|_{t=0} (\det \nabla(f+t\varphi))}{\sqrt{1+S(f)}}. \end{aligned} \tag{4.19}$$

Denote by $\nabla f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $\nabla\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have

$$\begin{aligned} \det\nabla(f+t\varphi) &= \det\begin{pmatrix} A+ta & B+tb \\ C+tc & D+td \end{pmatrix} \\ &= (A+ta)(D+td)-(B+tb)(C+tc) \\ &= \det\nabla f + t^2\det\nabla\varphi + t(aD-bC-cB+dA) \\ &= \det\nabla f + t^2\det\nabla\varphi + t < (\nabla f)^*, \nabla\varphi > . \end{aligned} \tag{4.20}$$

Therefore

$$\frac{d}{dt}\Big|_{t=0}(\det\nabla(f+t\varphi)) =< (\nabla f)^*, \nabla\varphi > . \tag{4.21}$$

Combining with (4.19), we get

$$\int_B < \frac{\nabla f + det(\nabla f)(\nabla f)^*}{\sqrt{1+S(f)}}, \nabla\varphi >= 0 \tag{4.22}$$

for any C^∞ function $\varphi : \overline{B} \to \mathbb{R}^2$ with $\varphi|_{\partial B} = 0$. Hence we have

$$div(\frac{\nabla f + det(\nabla f)(\nabla f)^*}{\sqrt{1+S(f)}}) = (0,0). \tag{4.23}$$

This means, f satisfies the elliptic system (4.23). Denote by $f = (u,v)$, with u, v two functions from $\overline{B}$ to $\mathbb{R}$. Denote by u_x, u_y, v_x, v_y the partial derivatives of f for short, and we write the system (4.23) in the standard non-linear form below

$$\begin{cases} \dfrac{\partial}{\partial x}[\dfrac{(1+v_y^2)u_x - (v_xv_y)u_y}{\sqrt{1+S(f)}}] + \dfrac{\partial}{\partial y}[\dfrac{(1+v_x^2)u_y - (v_xv_y)u_x}{\sqrt{1+S(f)}}] = 0, \\ \dfrac{\partial}{\partial x}[\dfrac{(1+u_y^2)v_x - (u_xu_y)v_y}{\sqrt{1+S(f)}}] + \dfrac{\partial}{\partial y}[\dfrac{(1+u_x^2)v_y - (u_xu_y)v_x}{\sqrt{1+S(f)}}] = 0. \end{cases} \tag{4.24}$$

Now set, for any $\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in M_2(\mathbb{R})$, $T\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} = 1 + a^2 + b^2 + c^2 + d^2 + (ad-bc)^2$, and

$$\begin{aligned} A_x^x\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} &= \frac{(1+d^2)a - bcd}{\sqrt{T\begin{pmatrix} a\ b \\ c\ d \end{pmatrix}}}, A_y^x\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} = \frac{(1+b^2)c - abd}{\sqrt{T\begin{pmatrix} a\ b \\ c\ d \end{pmatrix}}}, \\ A_x^y\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} &= \frac{(1+c^2)b - acd}{\sqrt{T\begin{pmatrix} a\ b \\ c\ d \end{pmatrix}}}, A_y^y\begin{pmatrix} a\ b \\ c\ d \end{pmatrix} = \frac{(1+a^2)d - abc}{\sqrt{T\begin{pmatrix} a\ b \\ c\ d \end{pmatrix}}}. \end{aligned} \tag{4.25}$$

Then these functions $A_i^j, i, j = x, y$ are C^∞ near the origin, and for any compact neighborhood K near the origin, all its derivatives are uniformly controlled by some constant depending on K.

The system (4.24) becomes

$$\begin{cases} D_x(A_x^x(\nabla f)) + D_y(A_y^x(\nabla f)) = 0, \\ D_x(A_x^y(\nabla f)) + D_y(A_y^y(\nabla f)) = 0. \end{cases} \tag{4.26}$$

We differentiate (4.26) with respect to x, we have

$$\begin{cases} D_x[D_aA_x^x(\nabla f)\cdot D_xu_x + D_bA_x^x(\nabla f)\cdot D_xv_x + D_cA_x^x(\nabla f)\cdot D_yu_x + D_dA_x^x(\nabla f)\cdot D_yv_x]+ \\ \quad D_y[D_aA_y^x(\nabla f)\cdot D_xu_x + D_bA_y^x(\nabla f)\cdot D_xv_x + D_cA_y^x(\nabla f)\cdot D_yu_x + D_dA_y^x(\nabla f)\cdot D_yv_x] = 0, \\ D_x[D_aA_x^y(\nabla f)\cdot D_xu_x + D_bA_x^y(\nabla f)\cdot D_xv_x + D_cA_x^y(\nabla f)\cdot D_yu_x + D_dA_x^y(\nabla f)\cdot D_yv_x]+ \\ \quad D_y[D_aA_y^y(\nabla f)\cdot D_xu_x + D_bA_y^y(\nabla f)\cdot D_xv_x + D_cA_y^y(\nabla f)\cdot D_yu_x + D_dA_y^y(\nabla f)\cdot D_yv_x] = 0. \end{cases} \tag{4.27}$$

This means that the function (u_x, v_x) satisfies the above system, with coefficient matrix

$$A(\nabla f) = \begin{pmatrix} D_aA_x^x(\nabla f) & D_cA_x^x(\nabla f) & D_aA_x^y(\nabla f) & D_cA_x^y(\nabla f) \\ D_aA_y^x(\nabla f) & D_cA_y^x(\nabla f) & D_aA_y^y(\nabla f) & D_cA_y^y(\nabla f) \\ D_bA_x^x(\nabla f) & D_dA_x^x(\nabla f) & D_bA_x^y(\nabla f) & D_dA_x^y(\nabla f) \\ D_bA_y^x(\nabla f) & D_dA_y^x(\nabla f) & D_bA_y^y(\nabla f) & D_dA_y^y(\nabla f) \end{pmatrix}. \tag{4.28}$$

We calculate the partial derivates of $A_i^j, i, j = x, y$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, and get

$$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1+d^2-(A_x^x)^2}{\sqrt{T}} & \frac{-A_x^xA_y^x-bd}{\sqrt{T}} & \frac{-A_x^xA_x^y-cd}{\sqrt{T}} & \frac{-A_x^yA_y^x+2bc-ad}{\sqrt{T}} \\ \frac{-A_y^xA_x^x-bd}{\sqrt{T}} & \frac{1+b^2-(A_y^x)^2}{\sqrt{T}} & \frac{-A_y^yA_x^x+2ad-bc}{\sqrt{T}} & \frac{-A_y^yA_y^x-ab}{\sqrt{T}} \\ \frac{-A_x^xA_x^y-cd}{\sqrt{T}} & \frac{-A_x^xA_y^y+2ad-bc}{\sqrt{T}} & \frac{1+c^2-(A_x^y)^2}{\sqrt{T}} & \frac{-A_x^yA_y^y-ac}{\sqrt{T}} \\ \frac{-A_y^xA_x^y+2bc-ad}{\sqrt{T}} & \frac{-A_y^xA_y^y-ab}{\sqrt{T}} & \frac{-A_y^yA_x^y-ac}{\sqrt{T}} & \frac{1+a^2-(A_y^y)^2}{\sqrt{T}} \end{pmatrix}. \tag{4.29}$$

We can observe that when a, b, c, d are small enough, $A\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the strong elliptic condition (3.12) in [7], hence the coefficient matrix $A(\nabla f)$ of (4.29) satisfies the strong elliptic condition, when μ is small. Moreover the $C^{0,\sigma}$ norm of $A(\nabla f)$ is also controlled by $||f||_{C^{1,\sigma}}$, and hence by μ.

Hence for the function (u_x, v_x), by Caccioppoli's inequality (cf.[7] Theorem 4.4), we have

$$||\nabla(u_x, v_x)||_{L^2(\overline{B}(0,\frac{7}{8}))} \leq C||(u_x, v_x)||_{L^2(\overline{B}(0,\frac{8}{9}))} \leq C||f||_{C^{1,\sigma}}, \tag{4.30}$$

where C depends on the $C^{0,\sigma}$ norm of the coefficient matrix $A(\nabla f)$, hence by $||f||_{C^{1,\sigma}}$, hence by μ.

Then by the Schauder estimates (Theorem 5.17 of [7]), we have

$$||\nabla(u_x, v_x)||_{C^{0,\sigma}(\overline{B}(0,\frac{6}{7}))} \leq C(\mu)||\nabla(u_x, v_x)||_{L^2(\overline{B}(0,\frac{7}{8}))} \leq C||f||_{C^{1,\sigma}} \leq C_2'(\mu), \tag{4.31}$$

where $C_2'(\mu) \to 0$ while $\mu \to 0$.

We differentiate the system (4.26) with respect to y, and get the same estimation

$$||\nabla(u_y, v_y)||_{C^{0,\sigma}(\overline{B}(0,\frac{6}{7}))} \leq C_2'(\mu). \tag{4.32}$$

Hence we get

$$||f||_{C^{2,\sigma}(\overline{B}(0,\frac{6}{7}))} \le C_2(\mu), \tag{4.33}$$

with $\lim_{\mu\to 0} C_2(\mu) = 0$.

We still need to estimate $\nabla^3 f$. For this we differentiate the system (4.27). We set $g_1 = u_x, g_2 = v_x$, and for $i = x, y, j = 1, 2$, set $p_{x1} = a, p_{x2} = b, p_{y1} = c, p_{y2} = d$. Then (4.27) becomes,

$$\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} A_\alpha^\beta(\nabla f)\cdot D_i g_j\Big) = 0, \text{ for } \beta = x, y. \tag{4.34}$$

Now we differentiate it with respect to s, for $s \in \{x, y\}$, and get

$$\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} A_\alpha^\beta(\nabla f)\cdot D_i(D_s g_j)\Big) + \sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} D_s A_\alpha^\beta(\nabla f)\cdot D_i g_j\Big) = 0, \tag{4.35}$$

$\beta = x, y$. I.e. the function $(D_s g_1, D_s g_2)$ satisfies the elliptic system

$$\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} A_\alpha^\beta(\nabla f)\cdot D_i(D_s g_j)\Big) = -\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} D_s A_\alpha^\beta(\nabla f)\cdot D_i g_j\Big). \tag{4.36}$$

Notice that the left hand side of the system is exactly the same as (4.34), hence the function $(D_s g_1, D_s g_2)$ is a solution to the elliptic system

$$\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} D_{P_{ij}} A_\alpha^\beta(\nabla f)\cdot D_i(D_s g_j)\Big) = -\sum_{\alpha=x,y} D_\alpha\Big(\sum_{i=x,y,j=1,2} B_{i,j}^{\alpha,\beta}\Big), \tag{4.37}$$

where $B_{i,j}^{\alpha,\beta} = D_{P_{ij}} D_s A_\alpha^\beta(\nabla f)\cdot D_i g_j$, hence $||B_{i,j}^{\alpha,\beta}||_{C^{0,\sigma}}$ is controlled by $||f||_{C^{2,\sigma}}$, which is controlled by $C_2(\mu)$, and is small.

We apply again the Caccioppoli's inequality for $(D_s g_1, D_s g_2)$, and get

$$\begin{aligned} ||\nabla(D_s g_1, D_s g_2)||_{L^2(\overline{B}(0,\frac{5}{6}))} &\le C\Big(||(D_s g_1, D_s g_2)||^2_{L^2(\overline{B}(0,\frac{7}{8})} + ||\sum_{\alpha=x,y,\beta=x,y}\ \sum_{i=x,y,j=1,2} B_{i,j}^{\alpha,\beta}||^2_{L^2(\overline{B}(0,\frac{6}{7})}\Big)^{\frac{1}{2}} \\ &\le C(||\nabla f||^2_{L^2(\overline{B}(0,\frac{7}{8})}) \le C_3'(\mu), \end{aligned} \tag{4.38}$$

with $\lim_{\mu\to 0} C_3'(\mu) = 0$.

Then we apply again the Schauder estimates (Theorem 5.17 of [7]), and get

$$\begin{aligned} ||\nabla(D_s g_1, D_s g_2)||_{C^{0,\sigma}(\overline{B}(0,\frac{4}{5}))} &\le C(||\nabla(D_s g_1, D_s g_2)||_{L^2(\overline{B}(0,\frac{5}{6}))} + ||\sum_{i=x,y,j=1,2} B_{i,j}^{\alpha,\beta}||_{C^{0,\sigma}(\overline{B}(0,\frac{6}{7}))}) \\ &\le C_3''(\mu), \text{ for } s = x, y, \end{aligned} \tag{4.39}$$

with $\lim_{\mu\to 0} C_3''(\mu) = 0$.

Recall that $(g_1, g_2) = (u_x, v_x)$. We repeat the same argument for (u_y, v_y), and altogether we have

$$||\nabla^3 f||_{C^{0,\sigma}(\overline{B}(0,\frac{4}{5}))} \le C_3(\mu), \tag{4.40}$$

with $\lim_{\mu\to 0} C_3(\mu) = 0$.

Combining (4.1), (4.15), (4.33) and (4.40), we have that for any μ small, there exists a constant

$C(\mu)$, with $\lim_{\mu\to 0} C(\mu) = 0$, such that if f is a minimal graph on $B(0,1)$, with

$$\max\{||f|_{\partial B(0,1)}||_{L^\infty}, ||Df|_{\partial B(0,1)}||_{L^\infty}\} \leq \mu, \tag{4.41}$$

then

$$\max_{0\leq i\leq 3} ||\nabla^i f||_{L^\infty(B(0,\frac{3}{4})}) \leq C(\mu). \tag{4.42}$$

Thus we complete the proof of Proposition 4.2. □

5 Estimates for perturbations around a minimal graph

Denote by $B = B(0,1) \cap \mathbb{R}^2$ the unit disc in $\mathbb{R}^2$. Let $q \in B(0, \frac{1}{100})$, and set $B_r = B(q,r)$ for $r > 0$. Fix any ϵ and l less than 10^{-4}, let $\mu < 10^{-4}$ be small. (Here in this section the three are independent; in the next section, l will be chosen first, and then ϵ will depend on l, and both will be fixed at the beginning, while μ will be supposed to be much smaller than these two, and will be decided later.) Let f be a function from $\overline{B}$ to $\mathbb{R}^2$ whose graph $\Sigma_f = \{(x, f(x)); x \in \overline{B}\} \subset \mathbb{R}^4$ is a minimal submanifold in $\mathbb{R}^4$, with $||f|_{\partial B}||_{C^1} \leq \mu$. Let h be a C^1 function from $A_r := \overline{B}\backslash B_r$ to $\mathbb{R}^2$ with $h|_{\partial B} = 0$, Lip $h \leq l$, and there exists a vector $M \in \mathbb{R}^2$ such that for any $x \in \partial B_r$, $|h(x) - M| \leq \epsilon r$. Denote by Σ_{f+h} the graph of $f + h$ on the annulus A_r.

Proposition 5.1. *Take all the notations and assumptions above, then*

$$H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \geq \frac{1}{4}\int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + C_0(\mu)), \tag{5.2}$$

where $\lim_{\mu\to 0} C_0(\mu) = 0$.

Proof. Now let us compare Σ_{f+h} and Σ_f above A_r. We have

$$\begin{aligned} H^2(\Sigma_{f+h}) - H^2(\Sigma_f) &= \int_{A_r} \sqrt{1 + S(f+h)} - \sqrt{1 + S(f)} \\ &= \int_{A_r} \sqrt{1 + S(f)}(\sqrt{\frac{1 + S(f+h)}{1 + S(f)}} - 1) \\ &= \int_{A_r} \sqrt{1 + S(f)}(\sqrt{1 + \frac{S(f+h) - S(f)}{1 + S(f)}} - 1). \end{aligned} \tag{5.3}$$

But

$$\begin{aligned} &S(f+h) - S(f) = [|\nabla(f+h)|^2 - |\nabla f|^2] + [(\det\nabla(f+h))^2 - (\det\nabla f)^2] \\ &= [2 < \nabla f, \nabla h > + |\nabla h|^2] + [< (\nabla f)^*, \nabla h > + \det\nabla h)][2\det\nabla f + \det\nabla h + < (\nabla f)^*, \nabla h >]. \end{aligned} \tag{5.4}$$

Notice that $|\nabla f| < 2\mu$, $|(\nabla f)^*| < 2\mu$ is small, and $|\det\nabla f| \leq |\nabla f|^2$, $|\det\nabla h| \leq |\nabla h|^2$, therefore $|S(f+h) - S(f)| < 1$ since $|\nabla h| < l$ is small. But $S(f) > 0$, hence $|\frac{S(f+h)-S(f)}{1+S(f)}| < 1$. For any $|x| < 1$ we have

$$1 + x = (1 + \frac{x}{2})^2 - \frac{x^2}{4} \geq (1 + \frac{x}{2} - \frac{x^2}{4})^2, \tag{5.5}$$

hence

$$\sqrt{1+\frac{S(f+h)-S(f)}{1+S(f)}} \geq 1+\frac{1}{2}\frac{S(f+h)-S(f)}{1+S(f)}-\frac{1}{4}(\frac{S(f+h)-S(f)}{1+S(f)})^2, \tag{5.6}$$

which gives

$$\begin{aligned} H^2(\Sigma_{f+h})-H^2(\Sigma_f) &\geq \int_{A_r}\sqrt{1+S(f)}(\frac{1}{2}\frac{S(f+h)-S(f)}{1+S(f)}-\frac{1}{4}(\frac{S(f+h)-S(f)}{1+S(f)})^2) \\ &= \frac{1}{2}\int_{A_r}\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}}-\frac{1}{4}\int_{A_r}\frac{(S(f+h)-S(f))^2}{(1+S(f))^{\frac{3}{2}}}. \end{aligned} \tag{5.7}$$

For the first term, by (5.4),

$$\begin{aligned} \frac{1}{2}\int_{A_r}\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}} &= \frac{1}{2}\int_{A_r}\frac{2<\nabla f,\nabla h>+|\nabla h|^2+2\det\nabla f<(\nabla f)^*,\nabla h>}{\sqrt{1+S(f)}}+ \\ &\frac{1}{2}\int_{A_r}\frac{2<(\nabla f)^*,\nabla h>\det\nabla h+<(\nabla f)^*,\nabla h>^2+2\det\nabla h\det\nabla f+|\det\nabla h|^2}{\sqrt{1+S(f)}} \\ &\geq \int_{A_r}\frac{<\nabla f,\nabla h>+\frac{1}{2}|\nabla h|^2+\det\nabla f<(\nabla f)^*,\nabla h>}{\sqrt{1+S(f)}}-(2\mu+l^2)\int_{A_r}|\nabla h|^2 \end{aligned} \tag{5.8}$$

But $S(f)\leq 5\mu^2$, hence $\frac{1}{1+S(f)}\geq\frac{8}{9}$, hence we have

$$\frac{1}{2}\int_{A_r}\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}}\geq\int_{A_r}<\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}},\nabla h>+\frac{1}{3}\int_{A_r}|\nabla h|^2. \tag{5.9}$$

By (4.23), and the hypothesis that $h|_{\partial B}=0$, we have

$$\begin{aligned} &\int_{A_r}<\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}},\nabla h> \\ &=\int_{\partial A_r}<h,[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>-\int_{A_r}<div(\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}),h> \\ &=-\int_{\partial B_r}<h,[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]> \\ &-\int_{\partial B_r}<(M+h-M),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]> \\ &=-<M,\int_{\partial B_r}[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>+\int_{\partial B_r}<(M-h),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>. \end{aligned} \tag{5.10}$$

For the second term of (5.10), since $|M-h|\leq\epsilon r$, Lip$f\leq\mu$, and $|det\nabla f|\leq 2|\nabla f|^2\leq 2\mu^2\leq\mu$ since μ is small, we have

$$|\int_{\partial B_r}<(M-h),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>|\leq\int_{\partial B_r}\epsilon r(2\mu)\leq 4\pi\mu\epsilon r^2. \tag{5.11}$$

For the first term of (5.10), first by Taylor expansion at the point 0, we have, for any $x\in\partial B_r$,

$$\nabla f(x)=\nabla f(0)+x\cdot\nabla^2 f(0)+o_1(r), \tag{5.12}$$

$$(\nabla f)^*(x)=(\nabla f)^*(0)+x\cdot\nabla(\nabla)^*f(0)+o_2(r), \tag{5.13}$$

$$\det(\nabla f)(x)=\det(\nabla f)(0)+x\cdot\nabla\det(\nabla f)(0)+o_3(r), \tag{5.14}$$

$$\frac{1}{\sqrt{1+S(f)}}(x)=\frac{1}{\sqrt{1+S(f)}}(0)+x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)+o_4(r) \tag{5.15}$$

where $|o_1(r)| \le r^2||\nabla^3 f||_{L^\infty(B(0,r))}$, $|o_2(r)| \le r^2||\nabla^3 f||_{L^\infty(B(0,r))}$, $|o_3(r)| \le r^2||\nabla^2 det(\nabla f)||_{L^\infty(B(0,r))}$, $|o_4(r)| \le r^2||\nabla^2(\frac{1}{\sqrt{1+S(f)}})||_{L^\infty(B(0,r))}$.

Hence we have

(5.16)

$$\begin{aligned}
&\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}=\\
&\{\nabla f(0)+x\cdot\nabla^2 f(0)+o_1(r)+[\det(\nabla f)(0)+x\cdot\nabla\det(\nabla f)(0)+o_3(r)][(\nabla f)^*(0)+x\cdot\nabla(\nabla)^*f(0)+o_2(r)]\}\\
&[\frac{1}{\sqrt{1+S(f)}}(0)+x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)+o_4(r)]\\
&=\{[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]+x\cdot[\nabla^2 f(0)+\nabla\det(\nabla f)(0)(\nabla f)^*(0)+\det(\nabla f)(0)\nabla(\nabla)^*f(0)]+o(r)\}\\
&[\frac{1}{\sqrt{1+S(f)}}(0)+x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)+o(r)]\\
&=[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)\\
&+x\cdot\frac{1}{\sqrt{1+S(f)}}(0)[\nabla^2 f(0)+\nabla\det(\nabla f)(0)(\nabla f)^*(0)+\det(\nabla f)(0)\nabla(\nabla)^*f(0)]\\
&+[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)][x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)]+o(r),
\end{aligned}$$

where all the $o(r)$ in (5.16) satisfied that $|o(r)| \le C_0 r^2$, where

$$C_0 = C(||\nabla f||_{L^\infty B(0,r)}, ||\nabla^2 f||_{L^\infty B(0,r)}, ||\nabla^3 f||_{L^\infty B(0,r)}) \tag{5.17}$$

tends to 0 as $||\nabla f||_{L^\infty B(0,r)}, ||\nabla^2 f||_{L^\infty B(0,r)}, ||\nabla^3 f||_{L^\infty B(0,r)}$ tend to 0.

Therefore,

(5.18)

$$\begin{aligned}
&|-<M,\int_{\partial B_r}[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>|\\
&\le|<M,\int_{\partial B_r}[\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)]>|\\
&+|<M,\int_{\partial B_r}[\vec{n}\cdot(x\cdot\frac{1}{\sqrt{1+S(f)}}(0)[\nabla^2 f(0)+\nabla\det(\nabla f)(0)(\nabla f)^*(0)+\det(\nabla f)(0)\nabla(\nabla)^*f(0)])]|\\
&+|<M,\int_{\partial B_r}\{\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)][x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)]\}>|+|<M,\int_{\partial B_r}o(r)>|.
\end{aligned}$$

For the first term of (5.18), since $[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)$ is a constant matrix, which we denote by V, and hence we have

$$<M,\int_{\partial B_r}\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)>=<M,(\int_{\partial B_r}\vec{n})\cdot V>=0 \tag{5.19}$$

because $\int_{\partial B_r}\vec{n}=0$.

For the second and third term of (5.16), notice that $|x| = r$, $\nabla f \leq \mu$, hence their sum is less than

$$C\mu r^2 + C|\nabla^2 f(0)|r^2 \leq (C\mu + CC_0)r^2, \tag{5.20}$$

where C_0 is as in (5.17) and C does not depend on μ, ϵ.

For the last, by the previous control on $o(r)$, this term is less than $C_0 r^3$.

Altogether we have

$$|- < M, \int_{\partial B_r} [\vec{n} \cdot \frac{\nabla f + \det \nabla f (\nabla f)^*}{\sqrt{1 + S(f)}}] > | \leq Cr^2(\mu + \ C_0). \tag{5.21}$$

Combining with (5.11) and (5.9), we have

$$\frac{1}{2} \int_{A_r} \frac{S(f+h) - S(f)}{\sqrt{1 + S(f)}} \geq \frac{1}{3} \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + \ C_0), \tag{5.22}$$

where C does not depend on μ, l and ϵ.

Recall that this is the estimation for the first term of the last line in (5.7). Now we treat its second term.

By (5.4), we have

$$\begin{aligned} &|S(f+h) - S(f)| \\ =&|[2 < \nabla f, \nabla h > + |\nabla h|^2] + [< (\nabla f)^*, \nabla h > + \det \nabla h)][2 \det \nabla f + \det \nabla h + < (\nabla f)^*, \nabla h >]| \\ \leq& 2|\nabla f||\nabla h| + |\nabla h|^2 + (|(\nabla f)^*||\nabla h| + |\nabla h|^2|][2|\nabla f|^2 + |\nabla h|^2 + |(\nabla f)^*||\nabla h|] \\ \leq& C(|\nabla f||\nabla h| + |\nabla h|^2) \leq C\mu|\nabla h| + C|\nabla h|^2, \end{aligned} \tag{5.23}$$

therefore the second term of (5.7) verifies

$$\begin{aligned} &-\frac{1}{4} \int_{A_r} \frac{(S(f+h) - S(f))^2}{(1 + S(f))^{\frac{3}{2}}} \geq -\frac{1}{4} \int_{A_r} (S(f+h) - S(f))^2 \\ &\geq -\frac{1}{4} \int_{A_r} (C\mu|\nabla h| + C|\nabla h|^2) \geq -C(\mu^2 + ||\nabla h||_\infty^2) \int_{A_r} |\nabla h|^2. \end{aligned} \tag{5.24}$$

On combining (5.7), (5.22) and (5.24) we get

$$\begin{aligned} H^2(\Sigma_{f+h}) - H^2(\Sigma_f) &\geq \frac{1}{3} \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + \ C_0) - C(\mu^2 + ||\nabla h||_\infty^2) \int_{A_r} |\nabla h|^2 \\ &\geq (\frac{1}{3} - C\mu^2 - Cl^2) \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + \ C_0). \end{aligned} \tag{5.25}$$

But Lip $h < l$ is small, hence we have

$$H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \geq \frac{1}{4} \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + C_0). \tag{5.26}$$

Now we apply Proposition 4.2, and get that when $r < \frac{3}{4}$ and μ is small enough,

$$C_0 = C_0(||\nabla f||_{L^\infty B(0,r)}, ||\nabla^2 f||_{L^\infty B(0,r)}, ||\nabla^3 f||_{L^\infty B(0,r)}) = C_0(C(\mu)) = C_0(\mu), \tag{5.27}$$

with $\lim_{\mu \to 0} C_0(\mu) = 0$. Thus we have

$$H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \geq \frac{1}{4} \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + C_0(\mu)). \tag{5.28}$$

$\square$

6 Conclusion

Now return to our set E. Recall that α is a pair of angles larger than $\theta_1' > \frac{\pi}{3}$. E is a reduced closed set that is minimal in $B(0,1)$, which contains no $2\mathbb{P}$ type point in $B(0,\frac{1}{100})$.

Set $l = 10^{-3}$, and suppose that $d_{0,1}^{\alpha} < \mu < \min\{\frac{\epsilon_0}{10}, \frac{l}{2}\}$, μ is to be decided later.

We apply Proposition 2.11 to E, with $\epsilon' = \min\{\epsilon_{\frac{l}{2}}, 10^{-4}\}$, (where $\epsilon_{\frac{l}{2}}$ corresponds to $\frac{l}{2}$ in Proposition 2.20), and get our o_E and r_E. Then $r_E < \frac{1}{4}$.

Let γ^i, g^i, as in Section 3. Suppose that

$$||\gamma^i||_{C^1} \leq \mu, i = 1, 2. \tag{6.1}$$

By Theorem 4.1 and 4.2 of [8], for each i there exists a function $f^i : \overline{B}(0,\frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$, whose graphs $\Sigma^i = \Sigma_{f^i} = \{(x, f(x)) : x \in \overline{B}(0,\frac{1}{2}) \cap P_\alpha^i\} \subset \mathbb{R}^4$ are minimal surfaces. Denote by $B^i(x,r) = B(x,r) \cap P_\alpha^i$.

On the other hand, we want to show the part of E in the annulus $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ is far from any translation of P_α. Recall that Proposition 2.11 says that E is $\epsilon' r_E$ far from any translation of P_α in the ball $D_\alpha(o_E, r_E)$. So for having a relatively big distance in the annulus, we simply use a compactness argument, and can get the following proposition. (See [9] for the proof).

Proposition 6.2 (cf.[9], Corollary 8.24). *For every $\epsilon > 0$, there exists $0 < \delta < \epsilon$, and $0 < \theta_0 < \frac{\pi}{2}$, which do not depend on ϵ, with the following properties. If $\theta_0 < \theta < \frac{\pi}{2}$, and if E is minimal in $D_\theta(0,1)$ and is δ near P_θ in $D_\theta(0,1) \backslash D_\theta(0,\frac{1}{4})$, and moreover*

$$p_\theta^i(E) \supset P_\theta^i \cap B(0, \frac{3}{4}), \tag{6.3}$$

then E is ϵ near P_θ in $D_\theta(0,1)$.

Let δ' be the δ corresponding to ϵ' in Proposition 6.2, we know that E is not $\delta' r_E$ near any translation of P_α in $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$. On the other hand, by definition of o_E and r_E, we know that the ϵ'-process does not stop at the scale $2r_E$, thus by Proposition 2.20, $E \cap D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ is composed of two fine C^1 graphs G^1, G^2 of two functions $g^i, i = 1, 2$ on $P_\alpha^i \cap D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ respectively. Thus $G^1 \cup G^2$ is not $\delta' r_E$ near any translation of P_α, there exists $i = 1, 2$ such that G^i is not δ' near any translation of P_α^i in $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$. Suppose this is the case for $i = 1$.

Denote by $g = g^1, f = f^1$, and $h = g - f$. We want to apply Proposition 5.1 to f and h, with $B(q,r) = B^1(o_E, \frac{1}{4} r_E)$ (hence $q = o_E, r = \frac{1}{4} r_E$). Recall that we have set $\epsilon' \leq \epsilon_{\frac{l}{2}}$, hence $|\nabla g|$ is smaller than $\frac{l}{2}$, which gives $|\nabla h| = |\nabla(g - f)|$ is smaller than $|\nabla g| + |\nabla f| < \frac{l}{2} + \mu < l$ cause μ is supposed to be less than $\frac{l}{2}$.

Also, by Proposition 2.11, G^1 is still $2\epsilon' r_E$ near some translation of P_α^1, hence there exists $M_g \in {P_\alpha^1}^\perp$ such that $|g(x) - M_g| \leq 2\epsilon' r_E = 8\epsilon' r$. But f is μ-Lipschitz, hence there exists M_f such that $|f(x) - M_f| \leq C\mu r$ on $\partial B(q,r)$, which gives $|h - (M_g + M_f)| \leq 9\epsilon' r < 10^{-3} r$ on $\partial B(q,r)$, when μ is small.

Now we can apply Proposition 5.1, and get

$$H^2(G^1) - H^2(\Sigma^1 \backslash C^1(o_E, \frac{1}{4}r_E)) = H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \geq \frac{1}{4}\int_{A_r} |\nabla h|^2 - Cr^2(\mu + \epsilon'\mu + C_0(\mu)), \tag{6.4}$$

with $A_r = B^1(0, \frac{1}{2}) \backslash B(q, r)$.

Now we want to estimate $\int_{A_r} |\nabla h|^2$. Recall that on $B^1(o_E, r_E) \backslash B^1(o_E, \frac{1}{4}r_E)$, the graph of g is $\delta' r_E$ far from any translation of P_α^1. On the other hand f is μ-Lipschitz, hence when μ is small, the graph of $h = g - f$ is $\frac{1}{2}\delta' r_E$ far from any translation of P_α^1.

Firstly we cite here two lemmas for estimating the Dirichlet's energy of our perturbation function h.

Lemma 6.5 (cf.[9], Corollary 7.23). *Let $r_0 > 0$, $q \in \mathbb{R}^2$ be such that $r_0 < \frac{1}{2}d(q, \partial B(0,1))$, suppose $u_0 \in C^1(\partial B(q, r_0) \cap \mathbb{R}^2, \mathbb{R})$, and denote by $m(u_0) = \frac{1}{2\pi r_0}\int_{\partial B(q, r_0)} u_0$ its average.*

Then for all $u \in C^1((\overline{B(0,1)} \backslash B(q, r_0)) \cap \mathbb{R}^2, \mathbb{R})$ that satisfies

$$u|_{\partial B(q, r_0)} = u_0 \tag{6.6}$$

we have

$$\int_{B(0,1) \backslash B(q, r_0)} |\nabla u|^2 \geq \frac{1}{4} r_0^{-1} \int_{\partial B(q, r_0)} |u_0 - m(u_0)|^2. \tag{6.7}$$

Lemma 6.8 (cf.[9], Corollary 7.36). *For all $0 < \epsilon < 1$, there exists $C = C(\epsilon) > 100$ such that if $0 < r_0 < 1$, $u \in C^1(\ B(0,1) \backslash B(0, r_0), \mathbb{R})$ and*

$$u|_{\partial B(0, r_0)} > \delta r_0 - \frac{\delta r_0}{C} \text{ and } u|_{\partial B(0,1)} < \frac{\delta r_0}{C} \tag{6.9}$$

then

$$\int_{B(0,1) \backslash B(0, r_0)} |\nabla u|^2 \geq \epsilon \frac{2\pi \delta^2 r_0^2}{|\log r_0|}. \tag{6.10}$$

Then denote by $P = P_\alpha^1$ for short. Denote by $D = D_\alpha$. Then h is a map from P to $P^\perp$, and is therefore from $\mathbb{R}^2$ to $\mathbb{R}^2$. Write $h = (\varphi_1, \varphi_2)$, where $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$. Then since the graph of h is $\frac{1}{2}\delta' r_E$ far from all translation of P, there exists $j \in \{1, 2\}$ such that

$$\sup_{x, y \in P \cap D(o_E, r_E) \backslash D(o_E, \frac{1}{4}r_E)} |\varphi_j(x) - \varphi_j(y)| \geq \frac{1}{4} r_E \delta'. \tag{6.11}$$

Suppose this is true for $j = 1$. Denote by

$$K = \{(z, \varphi_1(z)) : z \in (D(0, \frac{1}{2}) \backslash D(o_k, \frac{1}{4}r_E)) \cap P\}, \tag{6.12}$$

then

$$K \text{ is the orthogonal projection of } G^1 \cap D(0, \frac{1}{2}) \text{ on a 3-dimensional subspace of } \mathbb{R}^4. \tag{6.13}$$

For $\frac{1}{4}r_E \leq s \leq r_E$, define

$$\Gamma_s = K \cap p^{-1}(\partial D(o_E, s) \cap P) = \{(x, \varphi_1(x)) | x \in \partial D(o_E, s) \cap P\} \tag{6.14}$$

the graph of φ_1 on $\partial D(o_E, s) \cap P$.

We know that the graph of φ_1 is $\frac{1}{4}\delta' r_E$ far from P in $D(o_E, r_E)\backslash D(o_E, \frac{1}{4}r_E)$; then there are two cases:

1st case: there exists $t \in [\frac{1}{4}r_E, r_E]$ such that

$$\sup_{x,y\in\Gamma_t}\{|\varphi_1(x)-\varphi_1(y)|\} \geq \frac{\delta'}{C}r_E, \tag{6.15}$$

where $C = 4C(\frac{1}{2})$ is the constant of Lemma 6.8.

Then there exists $a, b \in \Gamma_t$ such that $|\varphi_1(a)-\varphi_1(b)| > \frac{\delta'}{C}r_E \geq \frac{\delta'}{C}t$. Since $||\nabla\varphi_1||_\infty \leq ||\nabla\varphi||_\infty < 1$, we have

$$\int_{\Gamma_t}|\varphi_1 - m(\varphi_1)|^2 \geq \frac{t^3\delta'^3}{4C^3} = (\frac{4}{3}t\delta')^3(\frac{27}{4^4C^3}). \tag{6.16}$$

Now in $D(0,\frac{1}{2})$ we have $d(0,o_E) < 6\epsilon' \leq 10\epsilon'\cdot\frac{1}{2}$, and $s < r_E < \frac{1}{8} < \frac{1}{2}\times\frac{1}{2}$, therefore we can apply Lemma 6.5 and obtain

$$\int_{(D(0,\frac{1}{2})\backslash D(o_E,t))\cap P}|\nabla\varphi_1|^2 \geq C(\delta')t^2 \geq C_1(\delta')r_E^2. \tag{6.17}$$

2nd case: for all $\frac{1}{4}r_E \leq s \leq r_E$,

$$\sup_{x,y\in\Gamma_s}\{|\varphi_1(x)-\varphi_1(y)|\} \leq \frac{\delta'}{C}r_E. \tag{6.18}$$

However, since

$$\begin{aligned}\frac{1}{2}r_E\delta' &\leq \sup\{|\varphi_1(x)-\varphi_2(y)| : x,y\in P\cap D(o_E,r_E)\backslash D(o_E,\frac{1}{4}r_E)\}\\ &= \sup\{|\varphi_1(x)-\varphi_2(y)| : s,s'\in[\frac{1}{4}r_E,r_E], x\in\Gamma_s, y\in\Gamma_{s'}\},\end{aligned} \tag{6.19}$$

there exist $\frac{1}{4}r_E \leq t < t' \leq r_E$ such that

$$\sup_{x\in\Gamma_t,y\in\Gamma_{t'}}\{|\varphi_1(x)-\varphi_1(y)|\} \geq \frac{1}{2}r_E\delta'. \tag{6.20}$$

Fix t and t', and without loss of generality, suppose that

$$\sup_{x\in\Gamma_t,y\in\Gamma_{t'}}\{\varphi_1(x)-\varphi_1(y)\} \geq \frac{1}{4}r_E\delta'. \tag{6.21}$$

Then

$$\inf_{x\in\Gamma_t}\varphi_1(x) - \sup_{x\in\Gamma_{t'}}\varphi_1(x) \geq \frac{1}{4}r_E\delta' - 2\frac{\delta'}{C}r_E = (1-\frac{2}{C(\frac{1}{2})})\frac{\delta'}{4}r_E \geq (1-\frac{2}{C(\frac{1}{2})})\frac{\delta'}{2}t' \tag{6.22}$$

because $C = 4C(\frac{1}{2})$.

Now look at what happens in the ball $D(o_E,t')\cap P$. Apply Lemma 6.8 to the scale t', we get

$$\int_{(D(o_E,t')\backslash D(o_E,t))\cap P}|\nabla\varphi_1|^2 \geq C(\delta',\frac{1}{2})\frac{\pi(\frac{\delta'}{2})^2t'^2}{\log\frac{t'}{t}}. \tag{6.23}$$

Then since $\frac{t'}{t} \leq 4, t' > t > \frac{1}{4}r_E$, we have

$$\int_{((D(o_E,t')\backslash D(o_E,t))\cap P}|\nabla\varphi_1|^2 \geq C_2(\delta')r_E^2. \tag{6.24}$$

So in both cases, there exists a constant $C = C_5(\delta') = \min\{C_1(\delta'), C_2(\delta')\}$, which depends only on

δ', such that

$$\int_{(D(0,\frac{1}{2})\backslash D(o_E,t_E))\cap P} |\nabla\varphi_1|^2 \geq C_5(\delta')r_E^2. \tag{6.25}$$

On the other hand, since $|\nabla\varphi_1| \leq |\nabla h| < 1$, we have

$$\int_{A_r} |\nabla h|^2 = \int_{(D(0,\frac{1}{2})\backslash D(o_E,t_E))\cap P} |\nabla h|^2 \geq C_5(\delta')r_E^2. \tag{6.26}$$

Thus by (6.4),

$$H^2(G^1) - H^2(\Sigma^1\backslash C^1(o_E, \frac{1}{4}r_E)) \geq C_5(\delta')r_E^2 - Cr_E^2(\mu + \epsilon'\mu + C_0(\mu)). \tag{6.27}$$

We apply also Proposition 5.1 to $i = 2$, where all the verifications for g^2, f^2, $h^2 = g^2 - f^2$ are similar to that of g^1, f^1, g^1. Hence we have

$$\begin{aligned} H^2(G^2) - H^2(\Sigma^2\backslash C^2(o_E, \frac{1}{4}r_E)) &\geq \frac{1}{4}\int_{P_\alpha^2\cap D(0,\frac{1}{2})\backslash D(o_E,\frac{1}{4}r_E)} |\nabla h|^2 - Cr_E^2(\mu + \epsilon'\mu + C_0(\mu)) \\ &\geq -Cr_E^2(\mu + \epsilon'\mu + C_0(\mu)). \end{aligned} \tag{6.28}$$

Now we still have to estimate the part inside $D(o_E, \frac{1}{4}r_E)$. For this purpose we need the following lemma.

Lemma 6.29 (cf.[9] Corollary 2.45)**.** *Suppose $\xi > 0$ is such that $\arccos(\xi/2) \leq \alpha_1 \leq \alpha_2$, and P^1, P^2 are two planes with characteristic angles (α_1, α_2). Denote by p^i the orthogonal projection on $P^i, i = 1, 2$. Then if E is a closed 2- rectifiable set satisfying $p^i(E) \supset B(0,1) \cap P^i$, we have*

$$H^2(E) \geq \frac{2\pi}{1+\xi}. \tag{6.30}$$

We apply Lemma 6.29 to the part $E \cap D_\alpha(o_E, \frac{1}{4}r_E)$, and by Proposition 2.28, we get

$$H^2(E \cap D_\alpha(o_E, \frac{1}{4}r_E)) \geq 2\pi(\frac{1}{4}r_E)^2\frac{1}{1 + 2\cos\theta_1'}. \tag{6.31}$$

On the other hand, notice that Lip $f^1 < C_0(\mu)$ and Lip $f^2 < C_0(\mu)$, we have

$$\begin{aligned} H^2(\Sigma^i \cap D_\alpha(o_E, \frac{1}{4}r_E)) &= \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)} \sqrt{1 + S(f)} \\ &\leq \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)} \sqrt{1 + C_0(\mu)^2 + C_0(\mu)^4} \\ &\leq \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)} 1 + \frac{C_0(\mu)^2 + C_0(\mu)^4}{2} \\ &= \pi(\frac{1}{4}r_E)^2(1 + \frac{C_0(\mu)^2 + C_0(\mu)^4}{2}), \end{aligned} \tag{6.32}$$

therefore

$$H^2(\Sigma \cap D_\alpha(o_E, \frac{1}{4}r_E)) \leq 2\pi(\frac{1}{4}r_E)^2(1 + \frac{C_0(\mu)^2 + C_0(\mu)^4}{2}). \tag{6.33}$$

Thus

$$H^2(\Sigma \cap D_\alpha(o_E, \frac{1}{4}r_E)) - H^2(E \cap D_\alpha(o_E, \frac{1}{4}r_E)) \leq 2\pi(\frac{1}{4}r_E)^2(\frac{C_0(\mu)^2 + C_0(\mu)^4}{2} + 2\cos\alpha_1). \tag{6.34}$$

We combine (6.34), (6.28) and (6.27), and get

$$
\begin{aligned}
&H^2(E\cap D(0,\frac{1}{2}))-H^2(\Sigma)\\
&=\sum_{i=1,2}[H^2(G^i-H^2(\Sigma^1\backslash C^1(o_E,\frac{1}{4}r_E))]+[H^2(E\cap D_\alpha(o_E,r_E))-H^2(\Sigma\cap D_\alpha(o_E,r_E)]\\
&\geq C_5(\delta')r_E^2-Cr_E^2(\mu+\epsilon'\mu+C_0(\mu))-Cr_E^2(\mu+\epsilon'\mu+C_0(\mu))\\
&\quad -2\pi(\frac{1}{4}r_E)^2(\frac{C_0(\mu)^2+C_0(\mu)^4}{2}+2\cos\alpha_1).
\end{aligned}
\tag{6.35}
$$

Notice that δ' is just a constant, depending on ϵ', where ϵ' is the parameter for the ϵ'-process, and guarantees the regularity for parts of minimal sets where the $\epsilon'-$process does not stop. Hence it does not depend on μ or α. Therefore when α is large enough and μ is small enough,

$$
H^2(E\cap D_\alpha(0,\frac{1}{2}))-H^2(\Sigma)>0. \tag{6.36}
$$

Recall that Σ contains a deformation of E in $D_\alpha(0,\frac{1}{2})$, hence (6.36) contradicts the fact that E is minimal.

This contradiction yields that there exists $\theta_1\in]0,\frac{\pi}{2}[$ and $\mu_0>0$ such that for any $\alpha>\theta_1$, if E is minimal in $B(0,1)$ with $d_{0,1}(E,P_\alpha)<\epsilon'$, and moreover (6.1) holds, then E contains a point of type $2\mathbb{P}$ in $B(0,\frac{1}{100})$.

Now for guarantee the condition (6.1), we apply Proposition 2.20 again. Set $\lambda=\epsilon_\mu$. Then when $d_{0,1}(E,P_\alpha)<\lambda$, our λ-process does not stop before step 1. Then by (2.22), the curves γ^i admits Lipschitz constants less than μ. Thus (6.1) holds.

Thus when $d_{0,1}(E,P_\alpha)\leq\lambda$, there exists a point of type $2\mathbb{P}$ in $B(0,\frac{1}{100})$. This completes the proof of Theorem 1.16. □

7 Global regularity and local C^1 regularity for minimal sets that are near $2\mathbb{P}$ type minimal cones

In this section we give two useful corollaries of Theorem 1.16, concerning global and local regularity for minimal sets that are near $2\mathbb{P}$ type minimal cones.

Theorem 7.1. *Let θ_1 be as in Theorem 1.16. Then for any $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\geq\alpha_1\geq\theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $\mathbb{R}^4$ such that one blow-in limit of E at infinity is P_α (i.e., there exists a sequence of numbers $r_n\to\infty$, and the sequence of sets $r_n^{-1}(E)$ converge to P_α under the Hausdorff distance as $n\to\infty$), then E is a $\mathbb{P}_\alpha$ set.*

Proof. By hypothesis, there exists $R>0$ and a $\mathbb{P}_\alpha$ set P_α such that $d_{0,R}(E,P_\alpha)<\lambda$. Then by Theorem 1.16, there exists a $2\mathbb{P}$ type point $x\in E$. In particular, the density $\theta(x)$ of E at x is 2, which is equal

to the density θ_∞ of E at infinity. By the monotonicity (cf.[5] Proposition 5.16) of the density function $\theta_x(r)=r^{-d}H^d(E\cap B(x,r))$, it has to be constant for $r\in]0,\infty[$. By Theorem 6.2 of [5], E is a minimal cone centered at x. As a result, $d_{x,r}(E,P_\alpha+x)$ is constant for $r\in]0,\infty[$, since $P_\alpha+x$ is also a cone centered at x. But by hypothesis, $d_{x,r}(E,P_\alpha+x)\to 0$ as $r\to\infty$, hence $d_{x,r}(E,P_\alpha+x)=0$, which means that $E=P_\alpha+x$. $\square$

Theorem 7.2. *Let θ_1 be as in Theorem 1.16. Then there exists a $\epsilon>0$ such that for any $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2\geq\alpha_1\geq\theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U\subset\mathbb{R}^4$, $B(x,100r)\subset U$, and there is a reduced minimal cone $P_\alpha+x$ of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,100r}(E,P_\alpha)\leq\epsilon$, then there exists a minimal cone $P_{\alpha'}$ of type $2\mathbb{P}$ such that there is a C^1 diffeomorphism $\Phi:B(x,2r)\to\Phi(B(x,2r))$, such that $|\Phi(y)-y|\leq 10^{-2}r$ for $y\in B(x,2r)$, and $E\cap B(x,r)=\Phi(P_{\alpha'})\cap B(x,r)$.*

Proof. Let λ be the λ in Theorem 1.16. Let $\epsilon=\min\{\frac{1}{1000}\lambda,\epsilon_1\}$, where ϵ_1 is the one in Corollary 12.25 of [6]. Then by Theorem 1.16, $d_{x,r}(E,P_\alpha)\leq 200d_{x,100r}(E,P_\alpha)\leq\frac{1}{5}\lambda$ yields that there exists a point $y\in B(x,\frac{1}{100}r)$ of type $P_{\alpha'}$ for some angle α'.

But $P_{\alpha'}\cap\partial B(0,1)$ is a disjoint union of two circles, and circles verifies the property of full length because of angles, hence by Remark 14.40 of [6], $P_{\alpha'}$ is a minimal cone with the full length property because of angles. We apply Theorem 1.15 of [6], and get the conclusion. $\square$

References

[1] William K. Allard. On the first variation of a varifold. *Ann.of Math.(2)*, 95:417–491, 1972.

[2] S. Bernstein. Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique. *Comm. Soc. Math. de Khardov*, 15:38–45, 1915-17.

[3] T. Carleman. Zur Theorie der Minimalflächen. *Math. Z.*, 9:154–160, 1921.

[4] Guy David. *Singular sets of minimizers for the Mumford-Shah functional.* Progress in Mathematics. Birkhäuser, 2005.

[5] Guy David. Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$. *Annales de la Faculté des Sciences de Toulouse*, XVIII(1):65–246, 2009.

[6] Guy David. $C^{1+\alpha}$-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$. *Journal of geometric analysis*, 20(4):837–954, 2010.

[7] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Edizioni della normale, 2005.

[8] Jr. H. B. Lawson and R. Osserman. Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system. *Acta Mathematica*, 139:1–17, 1977.

[9] Xiangyu Liang. Almgren-minimality of unions of two almost orthogonal planes in $\mathbb{R}^4$. *preprint, Orsay, arXiv:1103.1468*, 2011.

[10] Xiangyu Liang. Global regularity for minimal sets near a $\mathbb{T}$ set and counterexamples. *preprint, Orsay, arXiv: 1112.3565*, 2011.

[11] Xiangyu Liang. Almgren and topological minimality for the set $Y \times Y$. *preprint, Orsay, arXiv: 1203.0564*, 2012.

[12] Frank Morgan. Harnack type mass bounds and bernstein theorems for area-minimizing flat chains modulo ν. *Communications in Partial Differential Equations*, 11(12):1257–1283, 1986.

[13] Frank Morgan. Size-minimizing rectifiable currents. *Invent.math.*, 96:333–348, 1989.

[14] Jean Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math.(2)*, 103:489–539, 1976.